Families of Small Regular Graphs of Girth 5

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All graphs considered are finite, undirected and simple (without loops or multiple edges).

Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *girth* of a graph G is the length $g = g(G)$ of a shortest cycle. The *degree* of a vertex $v \in V$ is the number of vertices adjacent to v. A graph is called *k–regular* if all its vertices have the same degree k, and *bi–regular* or *(k_1, k_2)–regular* if all its vertices have either degree k_1 or k_2. A *(k, g)–graph* is a k–regular graph of girth g and a *(k, g)–cage* is a (k, g)–graph with the smallest possible number of vertices. The necessary condition obtained from the distance partition with respect to a vertex yields a lower bound $n_0(k, g)$ on the number of vertices of a (k, g)–graph, known as the Moore bound.

$$n_0(k,g) = \begin{cases} 1 + k + k(k-1) + \cdots + k(k-1)^{(g-3)/2} & \text{if } g \text{ is odd;} \\ 2(1 + (k-1) + \cdots + (k-1)^{g/2-1}) & \text{if } g \text{ is even.} \end{cases}$$

Biggs [8] calls *excess* of a (k, g)–graph G the difference $|V(G)| - n_0(k, g)$. Cages have been intensely studied since they were introduced by Tutte [27] in 1947. Erdős and Sachs [13] proved the existence of a (k, g)–graph for any value of k and g. Since then, most of the work carried out has been focused on constructing smallest (k, g)–graphs (see e.g. [1, 2, 3, 4, 5, 6, 7, 10, 14, 16, 19, 24, 25, 26, 28]). Biggs is the author of an impressive report on distinct methods for constructing cubic cages [9]. More details about constructions of cages can be found in the surveys by Wong [28], by Holton and Sheehan [21, Chapter 6], or the recent one by Exoo and Jajcay [15].

In this work we obtain $(q+3)$–regular graphs of girth 5 with fewer vertices than previously known ones (cf. [17, 22]) for $q = 13, 17, 19$ and for any prime $q \geq 23$ performing operations of reductions on the Levi graph B_q of an elliptic semiplane of type C (see [12, 18]) and then amalgams with bi–regular graphs into the obtained reduced graph or B_q itself. It is important to note that this graph B_q has also appeared in other different contexts (see e.g. [11, 20, 23]). We also obtain a new 13–regular graph of girth 5 on 236 vertices from B_{11} using the same technique.

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EXTENDED ABSTRACT:
An explicit formula for obtaining generalized quadrangles and others small regular graphs of girth 8

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1 Introduction

Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *girth* of a graph G is the number $g = g(G)$ of edges in a smallest cycle. For every $v \in V$, $N_G(v)$ denotes the *neighbourhood* of v, that is, the set of all vertices adjacent to v. The *degree* of a vertex $v \in V$ is the cardinality of $N_G(v)$. A graph is called *k-regular* if all the vertices have the same degree. A *cage* is a k-regular graph with girth g having the smallest possible number of vertices. Tutte [1] proved that a lower bound $n_0(k, g)$ on the number of vertices $n(k, g)$ in a cage is:

$$n_0(k,g) = \begin{cases} 1 + k + k(k-1) + \cdots + k(k-1)^{(g-3)/2} & \text{if } g \text{ is odd;} \\ 2(1 + (k-1) + \cdots + (k-1)^{g/2-1}) & \text{if } g \text{ is even.} \end{cases} \quad (1)$$

A (k, g)-cage with $n_0(k, g)$ vertices is called a *minimal cage*. The construction of graphs with small excess $n(k, g) - n_0(k, g)$ is a difficult task. Thus most of work carried out has focused on constructing a smallest one [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. Biggs is the author of a report on distinct methods for constructing cubic cages [17]. More details about constructions of cages can be found in the survey by Wong [16] or in the survey by Holton and Sheehan [18] or in the more recent dynamic cage survey by Exoo and Jajcay [19].

Minimal cages with even girth g exist only when $g \in \{4, 6, 8, 12\}$. If $g = 4$ they are the complete bipartite graph $K_{k,k}$, and for $g = 6, 8, 12$ they are the incidence graph of a generalized d-gon of order k. All these objets are known to exist for all prime power values of $k - 1$

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[20, 21, 22], and no example is known when $k-1$ is not a prime power. Thus, if q is a prime power, a $(q+1)$-regular graph with girth 8 and $n_0(q+1,8)$ vertices is the incidence graph of a generalized quadrangle of order q. Following this geometrical idea Benson [7] constructed minimal $(q+1,8)$-cages as follows. Let Q_4 be a non-degenerate quadric surface in projective 4-space $P(4,q)$. Define G_8 to be the graph whose nodes are the points and lines of Q_4, two nodes being joined if and only if they correspond to an incident point-line pair in Q_4. Then G_8 is a minimal $(q+1)$-regular graph of girth 8.

The first contribution of this work is a construction of these graphs in an alternative way by means of an explicit formula given in the following theorem.

Theorem 1.1 *Let $\mathbb{F}_q$ a finite field with $q \geq 2$ a prime power. Let $\Gamma_q = \Gamma_q[V_0, V_1]$ be a bipartite graph with $V_r = \{(a,b,c)_r, (q,q,a)_r : a \in \mathbb{F}_q \cup \{q\}, b, c \in \mathbb{F}_q\}$, $r = 0,1$. And the edge set of Γ_q is defined in the following way:*

For all $a \in \mathbb{F}_q \cup \{q\}$ and for all $b, c \in \mathbb{F}_q$:

$$N_{\Gamma_q}((a,b,c)_1) = \begin{cases} \{(j,\ aj+b,\ a^2j+2ab+c)_0 : j \in \mathbb{F}_q\} \cup \{(q,a,c)_0\} & \text{if } a \in \mathbb{F}_q; \\ \{(c,b,j)_0 : j \in \mathbb{F}_q\} \cup \{(q,q,c)_0\} & \text{if } a = q. \end{cases}$$

$$N_{\Gamma_q}((q,q,a)_1) = \{(q,a,j)_0 : j \in \mathbb{F}_q\} \cup \{(q,q,q)_0\}.$$

Then Γ_q is a $(q+1;8)$-cage on $2q^3+2q^2+2q+2$ vertices.

Remark 1.1 *(i) Let Γ_q be a $(q+1;8)$-cage obtained in Theorem 1.1. Using geometrical terminology we call the elements of V_0* lines *and the elements of V_0* points. *Then Γ_q is the incidence graph of a classical generalized quadrangle $Q(4,q)$.*

(ii) The edge set of a $(q+1;8)$-cage Γ_q obtained in Theorem 1.1 can equivalently be expressed as follows:

For all $x \in \mathbb{F}_q \cup \{q\}$ and for all $y, z \in \mathbb{F}_q$:

$$N_{\Gamma_q}((x,y,z)_0) = \begin{cases} \{(a,\ y-ax,\ a^2x-2ay+z)_1 : a \in \mathbb{F}_q\} \cup \{(q,y,x)_1\} & \text{if } x \in \mathbb{F}_q; \\ \{(y,a,z)_1 : a \in \mathbb{F}_q\} \cup \{(q,q,y)_1\} & \text{if } x = q. \end{cases}$$

$$N_{\Gamma_q}((q,q,z)_0) = \{(q,a,z)_1 : a \in \mathbb{F}_q\} \cup \{(q,q,q)_1\};$$
$$N_{\Gamma_q}((q,q,q)_0) = \{(q,q,x)_1 : x \in \mathbb{F}_q \cup \{q\}\}.$$

Therefore, if q is even, $2a = 0$ for all $a \in \mathbb{F}_q$ so that the partite sets V_0 and V_1 can be interchanged obtaining the same graph Γ_q. Equivalently, if q is even (in geometrical terminology) the corresponding generalized quadrangle $Q(4,q)$ is said to be self-dual.

In what follows we construct $(k,8)$-regular balanced bipartite graphs for $k = q-1, q$ where q is a prime powers q with order as small as possible. We will use the following notation. Given an integer $k \geq 1$, a graph G and a vertex $u \in V(G)$, let $N^k_G(u) = \{x \in V(G) : d_G(u,x) = k\}$, and $N^k_G[u] = \{x \in V(G) : d_G(u,x) \leq k\}$, where $d_G(u,x)$ denotes the distance between u and x in G.

A subset $U \subset V(G)$ is said to be *a fair dominating set of* G if for each vertex $x \in V(G) \setminus U$, $|N_G(x) \cap U| = 1$. Let $\Gamma_q = \Gamma_q[V_0, V_1]$ be the $(q+1, 8)$-cage constructed in Theorem 1.1. Suppose that U is a fair dominating set of Γ_q, then $\Gamma_q - U$ is a q-regular graph of girth 8. Thus it is of interest to find the largest fair dominating set of Γ_q. In the following theorem we find fair dominating sets of orders $2(q^2+1)$, $2(q^2+3q+1)$ and $2(q^2+4q+3)$ if q is even.

Theorem 1.2 *Let $q \geq 2$ be a prime power and $\Gamma_q = \Gamma_q[V_0, V_1]$ the $(q+1, 8)$-cage constructed in Theorem 1.1. The following sets are fair dominating in Γ_q:*

(i) $A = N^2_{\Gamma_q}[\alpha] \cup N^2_{\Gamma_q}[\beta]$ *where* $\alpha, \beta \in V(\Gamma_q)$ *and* $\beta \in N^3_{\Gamma_q}(\alpha)$. *Further* $|A| = 2(q+1)^2$.

(ii) $$B = \bigcup_{c \in \mathbb{F}_q} N_{\Gamma_q}[(q,0,c)_1] \cup N_{\Gamma_q}[(q,q,0)_1] \cup \left(\bigcap_{c \in \mathbb{F}_q} N^2_{\Gamma_q}[(q,0,c)_1] \cap N^2_{\Gamma_q}[(q,q,0)_1] \right) \cup N^2_{\Gamma_q}[(q,q,\xi)_1],$$
where $\xi \in \mathbb{F}_q \setminus \{0\}$. *Further* $|B| = 2(q^2+3q+1)$.

(iii)

$$\begin{aligned} C &= \bigcup_{x \in \mathbb{F}_q \cup \{q\}} N_{\Gamma_q}[(q,x,0)_0] \cup \left(\bigcap_{x \in \mathbb{F}_q \cup \{q\}} N^2_{\Gamma_q}[(q,x,0)_0] \right) \cup \bigcup_{x \in \mathbb{F}_q} N_{\Gamma_q}[(x,x,p(x))_1] \\ &\quad \cup N_{\Gamma_q}[(q,1,1)_1] \cup \left(\bigcap_{x \in \mathbb{F}_q} N^2_{\Gamma_q}[(x,x,p(x))_1] \cap N^2_{\Gamma_q}[(q,1,1)_1] \right), \end{aligned}$$

where $q \geq 8$ *is even and* $p(x) = 1 + x + x^2$ *for all* $x \in \mathbb{F}_q$. *Further* $|C| = 2(q^2+4q+3)$.

The fair dominating sets described in item *(ii)* and *(iii)* of Theorem 1.2 are depicted in Figure 1 and in Figure 2 respectively.

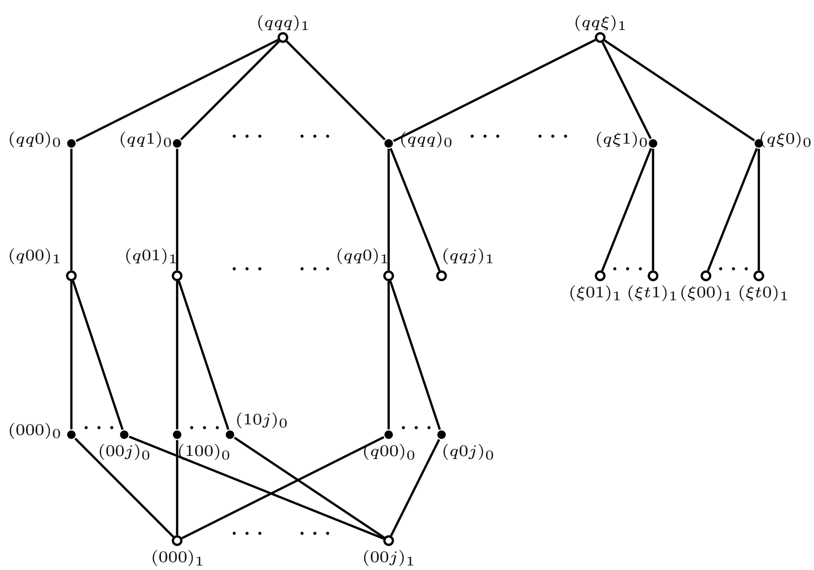

Figure 1: Deleted subgraph in *(ii)* of Theorem 1.3.

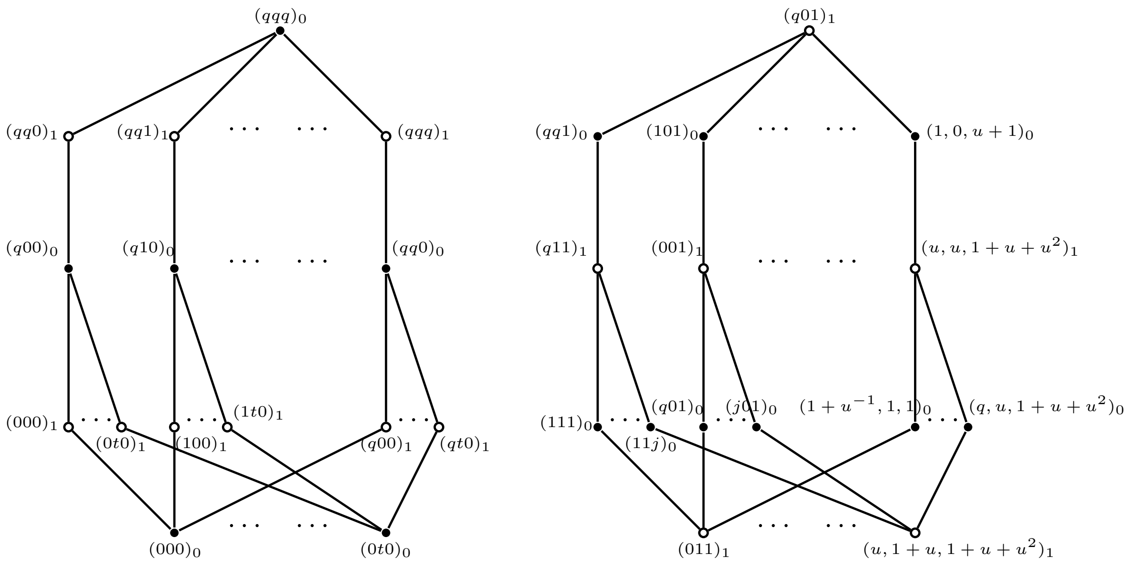

Figure 2: Deleted subgraph in *(iii)* of Theorem 1.3.

When $q = 2$ a cycle of length 8 is obtained by eliminating from Γ_2 the vertices indicated in Theorem 1.2 *(ii)*. And for $q = 2$ the $(3, 8)$-cage can be partitioned into two induced subgraphs indicated in Figure 2. For $q = 4$ the $(5, 8)$-cage also contains a fair dominating set as indicated in Theorem 1.2 (iii), which is as follows:

$$C = \bigcup_{x\in\mathbb{F}_4} N_{\Gamma_4}[(4,x,\xi)_0] \cup N_{\Gamma_4}[(4,4,0)_0] \cup \left(\bigcap_{x\in\mathbb{F}_4\cup\{4\}} N^2_{\Gamma_4}[(4,x,\xi)_0] \cap N_{\Gamma_4}[(4,4,0)_0] \right)$$

$$\bigcup_{x\in\mathbb{F}_4} N_{\Gamma_4}[(x,x,p(x))_1] \cup N_{\Gamma_4}[\{(4,1,1)_1\}] \cup \left(\bigcap_{x\in\mathbb{F}_4} N^2_{\Gamma_4}[(x,x,p(x))_1] \cap N^2_{\Gamma_4}[\{(4,1,1)_1\}] \right),$$

where $\xi \in \mathbb{F}_4 \setminus \{0, 1\}$, because in this case $p(x) = 1 + x + x^2 \in \{0, 1\}$ for all $x \in \mathbb{F}_4$.

For all $q \geq 2$ the following result is immediate as a consequence of Theorem 1.2.

Theorem 1.3 *Let $q \geq 2$ be a prime power and $\Gamma_q = \Gamma_q[V_0, V_1]$ the $(q+1, 8)$-cage constructed in Theorem 1.1. By removing from Γ_q a fair dominating set, q-regular graphs of girth 8 are obtained with orders $2q(q^2 - 1)$, $2q(q^2 - 2)$ or $2(q^3 - 3q - 2)$ if $q \geq 4$ is even.*

By using geometrical techniques, q-regular bipartite graphs of girth 8 on $2q(q^2 - 2)$ vertices if q is odd, or on $2(q^3 - 3q - 2)$ vertices if q is even, are given in [23]. Also using geometrical techniques $(q-1)$-regular small graphs on $2(q^3 - q^2 - q + 1)$ vertices have been obtained in [24]. And $(k, 8)$-regular balanced bipartite graphs for all prime powers q such that $3 \leq k \leq q$ of order $2k(q^2 - 1)$ have been obtained as subgraphs of the incidence graph of a generalized quadrangle [3]. This result has been improved by constructing $(k, 8)$-regular balanced bipartite graphs of order $2q(kq - 1)$ [6].

In the following theorem we improve this result for the case $k = q - 1$. Given a subset of vertices $S \in V(G)$ we denote by $N_G(S) = \cup_{s\in S} N_G(s)$.

Theorem 1.4 *Let $q \geq 4$ be a prime power and G_q the q-regular graph of girth 8 constructed in Theorem 1.3 on $2q(q^2-2)$ vertices choosing $\xi \in \mathbb{F}_q \setminus \{0,1\}$. Define $R = N_{G_q}(\{(q,y,z)_0 : y,z \in \mathbb{F}_q, y \neq 0,1,\xi\}) \cap N^5((q,1,0)_0)$. The following set is fair dominating in G_q:*

$$\bigcup_{z\in\mathbb{F}_q} N_{G_q}[(q,1,z)_0] \cup N_{G_q}[R].$$

Therefore a $(q-1)$-regular graph of girth 8 with $2q(q-1)^2$ vertices can be obtained by deleting from G_q the indicated fair dominating set.

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Necessary Conditions for (d, k)-digraphs

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Let (d, k)-*digraph* be a diregular digraph of degree $d \geq 2$, diameter $k \geq 2$ and order $d + d^2 + ... + d^k$, one less than the Moore bound. Such a (d, k)-digraph is also called an *almost Moore digraph*. In this (d, k)-digraph, for every vertex u there exists exactly one vertex v such that there are two walks of length $\leq k$ from u to v. The vertex v is called the *repeat* of u, denoted by $r(u) = v$. In case $r(u) = u$, vertex u is called a *selfrepeat* (the two walks, in this case, have lengths 0 and k).

The study of the existence of an almost Moore digraphs of degree d and diameter k has received much attention. Fiol, Allegre and Yebra (1983) showed the existence of $(d, 2)$-digraphs for all $d \geq 2$. In particular, for $d = 2$ and $k = 2$, Miller and Fris (1988) enumerated the exact number of $(2, 2)$-digraphs. Furthermore, Gimbert (2001) showed that there is only one $(d, 2)$-digraph, namely the line digraph $L(K_{d+1})$ of the complete digraph K_{d+1}, for $d \geq 3$. However for degree 2 and diameter $k \geq 3$, it is known that there is no $(2, k)$-digraph (Miller and Fris, 1992). Furthermore, it was proved that there is no $(3, k)$-digraph with $k \geq 3$ (Baskoro, Miller, Siran and Sutton, 2005). Thus, the remaining case still open is the existence of (d, k)-digraphs with $d \geq 4$ and $k \geq 3$.

Several necessary conditions for the existence of (d, k)-digraphs, for $d \geq 4$ and $k \geq 3$, have been obtained. In this talk, we shall discuss some necessary conditions for these (d, k)-digraphs. Open problems related to this study are also presented.

Regular automorphism group of 1-factorization of complete multipartite graphs

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A $1-$factor of a graph Γ is a spanning subgraph with all vertices of degree 1. A $1-$factorization of a graph Γ is a partition of its edges in disjoint $1-$factors. An automorphism group of a $1-$factorization of a graph G is a group of permutations of the vertices of Γ that map factors to factors. If the automorphism group of a $1-$factorization is sharply-transitive on its action on the vertices, the group is called regular. The problem of existence of a $1-$factorization of a graph Γ with a regular automorphism group has been considered for various types of graphs and groups. The benchmark of this topic is due to Hartman and Rosa [4] :

Theorem 1 *[4] A complete graph K_n admits a $1-$factorization with a cyclic automorphism group acting sharply transitively on the vertices if and only if n is even and* $n \neq 2^t$, $t \geq 3$.

This theorem has been generalized to the entire class of abelian groups [3] and to the class of finitely generated abelian groups [2]. In [1], the autors extended the problem to dihedral groups, leading to the following result.

Theorem 2 *[1] A complete graph K_n admits a $1-$factorization with a dihedral automorphism group acting sharply transitively on the vertices for all n even.*

The problem naturally extends to the class of complete multipartite graphs $K_{m\times n}$, as introduced in [5]. It is proved in [5] that the existence of a $1-$factorization of $K_{m\times n}$ having an automorphism group G acting sharply transitively on the vertices is equivalent to the existence of a particular starter, a very slight generalization of the concept of starter introduced by Buratti in [3] for the complete graph. We will present our results (joint work with G. Mazzuoccolo) and give new results for the existence of $1-$factorizations of $K_{m\times n}$ admitting a regular, cyclic or abelian automorphism group.

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Directed Cages and the Caccetta–Häggkvist Conjecture

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Extended Abstract

1 Directed Cages

A *cage* is a smallest regular graph of specified degree and girth. If the degree is d and the girth g, such a graph is called a (d, g)*-cage*. By considering a breadth-first search tree, the number n of vertices in a (d, g)-cage is easily seen to satisfy the lower bound:

$$n \geq \begin{cases} (2(d-1)^k - 2)/(d-2) & \text{if } g = 2k, \\ (d(d-1)^k - 2)/(d-2) & \text{if } g = 2k+1. \end{cases}$$

For $g \geq 5$, equality is attained only rarely. For instance, when $g = 5$, only three such graphs are known (of degrees $d = 2, 3, 7$); moreover, any further candidate must be of degree 57. The methods used to study cages are primarily algebraic (see, for example, the survey by Wong [?]).

The situation with regard to directed graphs is quite different. For simplicity, *the digraphs discussed here contain no cycles of length less than three (directed or otherwise).* We shall call a digraph d*-regular* of *girth* g if each vertex has indegree and outdegree d and if the shortest directed cycle is of length g. A *directed* (d, g)*-cage* is thus a smallest d-regular digraph of girth g.

As before, there is a simple bound for the number n of vertices in such a digraph. However this time it is an upper bound, provided by the d-regular digraph $\overrightarrow{C_n}^d$, the d-th power of the directed cycle $\overrightarrow{C_n}$. If $n = (g-1)d + 1$, then $\overrightarrow{C_n}^d$ has girth g, yielding the bound

$$n \leq (g-1)d + 1.$$

Behzad, Chartrand and Wall [?] conjectured that equality holds. An equivalent formulation of their conjecture is as follows.

Behzad–Chartrand–Wall Conjecture. *Let D be a d-regular digraph on n vertices. Then the girth of D is at most $\lceil n/d \rceil$.*

Powers of directed cycles are not the only hypothetical extremal digraphs for this conjecture.

The lexicographic product of two 'extremal' digraphs of the same girth yields yet another one. The Behzad–Chartrand–Wall Conjecture has been confirmed for small values of d by various authors (see below), and for vertex-transitive digraphs, by Hamidoune [?]. A proof for vertex-transitive digraphs can also be found in an expository article by Nathanson [?].

2 The Caccetta–Häggkvist Conjecture

Caccetta and Häggkvist [?] proposed a generalization of the Behzad–Chartrand–Wall Conjecture, requiring only a lower bound on the outdegrees of the digraph.

Caccetta–Häggkvist Conjecture. *Let D be a digraph on n vertices in which each vertex is of outdegree at least d. Then the girth of D is at most $\lceil n/d \rceil$.*

Remark. A digraph in which each vertex is of outdegree at least d contains a spanning subgraph in which each vertex is of outdegree *exactly* d, so the qualification 'at least' in the statement of the Caccetta-Häggkvist conjecture is superfluous.

This conjecture is more amenable to inductive arguments. It holds trivially for $d = 1$, and has been verified for several other small values of d: by Caccetta and Häggkvist [?]) for $d = 2$, by Hamidoune [?] for $d = 3$, and by Hoàng and Reed [?] for $d = 4, 5$. More generally, Shen [?] has shown that it holds whenever $n \geq 2d^2 - 3d + 1$.

Chvátal and Szemerédi [?] established the bounds $2n/(d + 1)$ and $n/d + 2500$ on the directed girth. Nishimura [?] reduced the additive constant in the latter bound to 304 and Shen [?] reduced it further to 73. These results are asymptotically best possible for $d = o(n)$. However, they are far from tight when $d = \lceil cn \rceil$ with $c > 0$. A particular effort has been expended on the case $c = 1/3$. This instance of the Caccetta–Häggkvist Conjecture can be rephrased as:

Directed Triangle Conjecture. *Let D be a digraph on n vertices in which each vertex is of outdegree $\lceil n/3 \rceil$. Then D contains a directed triangle.*

It is perhaps surprising that this special case is still open. Short of proving the conjecture, one may seek as small a value of c as possible such that every digraph on n vertices with minimum outdegree at least cn contains a triangle. This was the strategy of Caccetta and Häggkvist [?], who obtained the value $c = (3 - \sqrt{5})/2 \approx 0.3820$ by a simple inductive argument. That bound was improved successively to 0.3798 by Bondy [?], 0.3542 by Shen [?], 0.3532 by Hamburger, Haxell and Kostochka [?], and 0.3465 by Hladký, Král, and Norine [?] using the 'flag algebras' introduced by Razborov [?], [?]. Various equivalent formulations of the Directed Triangle Conjecture are discussed by Charbit [?].

A conjecture intermediate between the Behzad–Chartrand–Wall and Caccetta–Häggkvist conjectures was considered by de Graaf, Schrijver, and Seymour [?]. They proved that any digraph on n vertices with both minimum indegree and minimum outdegree at least cn, where $c \approx 0.3488$, contains a triangle, thereby strengthening and extending a bound for regular digraphs found by Li and Brualdi [?]. This bound has also seen several successive improvements, by de Graaf [?] to 0.3461, by Hamburger, Haxell, and Kostochka [?] to 0.3457, and most recently to 0.3436 by Lichiardopol [?], using the result of Hladký, Král, and Norine cited above.

3 Related Questions

One sign of a good problem is that it gives rise to further good problems. We mention two here.

Seymour [?] proposed the following beautiful conjecture, which would imply the triangle case of the Behzad–Chartrand–Wall Conjecture. A *second outneighbour* of a vertex v in a digraph is a vertex whose distance from v is exactly two.

Second Neighbourhood Conjecture. *Every digraph has a vertex v with at least as many second outneighbours as outneighbours.*

The Second Neighbourhood Conjecture was verified for tournaments by Fisher [?], and later by Havet and Thomassé [?], using quite different methods. This special case of the conjecture is due to Dean and Latka [?].

The Caccetta–Häggkvist Conjecture relates three parameters, the order, the directed girth and the minimum outdegree. A strengthening which brings into play the structure of the digraph was formulated by Hoàng and Reed [?].

Hoàng–Reed Conjecture. *Let D be a digraph in which each vertex is of outdegree d. Then D contains d directed cycles $C_1, \ldots, C_d$ such that C_j meets $\cup_{i=1}^{j-1} C_i$ in at most one vertex, $1 < j \leq d$.*

In other words, D contains a 'forest' of d arc-disjoint directed cycles, any two meeting in at most one vertex. In this case, a simple calculation shows that one of the d directed cycles has length at most $\lceil n/d \rceil$.

The Hoàng–Reed Conjecture holds for $d = 2$ by a result of Thomassen [?], and was verified for $d = 3$ by Welhan [?]. Havet, Thomassé and Yeo [?] showed that it is true also for tournaments.

One may formulate variants and refinements of the Hoàng–Reed Conjecture by considering special forests of directed cycles, such as stars and paths. For instance, Seymour [?] asked whether any d-regular digraph has a vertex v and d arc-disjoint directed cycles meeting only at v. This is true for $d = 3$ by a theorem of Thomassen [?]. However, Mader [?] showed that it is false for all $d \geq 8$. On the other hand, Mader [?] proved that there is always such a star of directed cycles in a vertex-transitive digraph. Hamidoune [?] gave a simpler proof of this fact. Mader [?] also found examples, for all $d \geq 4$, of d-regular vertex-transitive digraphs containing no path of d arc-disjoint directed cycles.

For an extensive survey on the topic, with a multitude of related open problems, see Sullivan [?].

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Generating regular directed graphs

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Abstract

In this talk we describe an algorithm to efficiently generate all regular directed graphs for a given number of vertices and given degree. So far the fastest way to construct regular directed graphs was by assigning directions to the underlying undirected graphs. Efficient programs for this task are e.g. *directg* [2] or *water* [1]. In this talk, the directed graphs are constructed from bipartite (undirected) graphs by forming pairs of vertices with one vertex in every class.

The bijective correspondence between (a) directed graphs with in- and out-degree k and n vertices and (b) k-regular bipartite 2-coloured (say black and white) graphs with $2n$ vertices and a fixed perfect matching of the **complement** is simple:

For a k-regular directed graph D the corresponding bipartite graph is formed by replacing every vertex v by two vertices – a black vertex v_b and a white vertex v_W. Then for every directed edge $a \to b$ an undirected edge $\{a_b, b_w\}$ is inserted. The matching of the complement is the set of all $\{v_b, v_w\}$ with v a vertex of the directed graph.

The reverse operation is to direct all edges from the black to the white bipartition and identify vertices as described by the given matching. This operation guarantees that no loops or double edges will be present in the directed graphs, but without further restrictions to the matching, oppositely directed edges with the same pair of endpoints may occur – and unless explicitly stated otherwise that is also the intention.

Bipartite regular graphs can be efficiently generated by Markus Meringer's program *genreg* [3], so what remains to be done is to develop methods to generate all non-equivalent matchings of the complement of a regular bipartite graph – that is: matchings that lead to non-isomorphic directed graphs. This will be the main topic of this talk. The basic method applied is McKay's

canonical construction path methode described in [2] – but with some new – and hopefully in an more general context applicable – additional optimisations.

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Extended abstract

From Groups to graphs in the footsteps of Jacques Tits

1. The Abel Prize 2008 was awarded to Jacques Tits and John Thompson. The motivation concerning Tits was mainly based on his Theory of Buildings. This theory ought to be a chapter in Graph Theory. Let us restrict to spherical buildings which include all finite buildings.
Such a building is a graph submitted to conditions, no more, no less.
Also, most such buildings have a large automorphism group whose action is transitive on ordered maximal cliques. Knowing the abstract group allows for a uniform construction of all buildings (of rank >2).

2.Jacques Tits, buildings and graphs
Born: Uccle (Brussels) 1930
Belgian then French in 1974
Winner of Abel Prize in 2008 for his fundamental Theory of BUILDINGS spread over 1954-2004 with birth of Buildings in 1961 and Theory of Spherical Buildings completed in 1974.

Buildings are graphs !!!

3.Buildings as graphs
3.1. Restrict to spherical buildings namely those all of whose apartments are finite.
We start defining a spherical building SB.

3.2. SB is a graph (undirected, no loops)
AXIOM: SB is multipartite of some rank $r > 0$.
AXIOM: Every maximal clique has r vertices
3.3. DEF: SB is THICK if every comaximal clique is in at least three maximal ones.
DEF: SB is THIN if every comaximal clique is in exactly two maximal ones.
3.4. NOTATION: For any set V of vertices, V^* is the set of vertices not in V that are adjacent to all in V.
AXIOM: For every clique c of rank in [0, r -2], c^* is CONNECTED.

4.Apartments
4.5. SB is equipped with a family A of subgraphs of rank r called apartments
AXIOM: every apartment is thin
4.6. AXIOM: In SB, any two cliques are contained in some apartment (BNB property)
4.7. AXIOM: In SB, for any two apartments U and V and cliques C,D in their intersection there exists a (type preserving) isomorphism of U onto V fixing C and D vertexwise ("germ" of group)
4.8 A fantastic Theory is developed on this basis
Examples that were analyzed for years because of the complexity they entail, come from the Theory of Lie-Chevalley groups, without any case analysis.
A major result : every thick spherical building of rank $r>2$ is one of the Lie-Chevalley examples.

5. From groups other than Lie-Chevalley to graphs
-Mostly one of the 26 sporadic simple groups.
Has been a project of mine since 1975.
Great advances made by various persons in particular Dimitri Leemans.
-Still other groups met in the study of sporadic groups.
-First of all, we look at the Golay code with no attention for Coding Theory.

6.Coset graph of the extended binary Golay code
Group $2^{12} : M_{24}$
distance-transitive
Vertex stabilizer M_{24}
Graph on 4096 vertices,bipartite (2048+2048), degree 24, girth 4,diameter 4.

7. Janko group J_4

86 775 571 046 077 562 880 approx $8:10^{19}$
$2^{21}\ 3^5\ 5\ 7\ 11\ 23\ 29\ 31\ 37\ 43$
Predicted by Z.Janko 1975
Existence with help of a computer: S.Norton, C.Parker, J.Thackray, D.J.Benson, J.H.Conway 1980
$J_4 < GL_{112}(F_2)$
Computer free: A.A. Ivanov and U. Meierfrankenfeld 1999
Picture of SMALLEST GRAPH on which J_4 acts transitively.
It has 173 067 389 vertices and degree 15180
The graph is due to Peter Rowley and Louise Walker 1994
The stabilizer of a vertex is $2^{12} : M_{24}$

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The Number of Spanning Trees of the Hanoi Graph

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The problem of finding the number of spanning trees of a finite graph is a relevant and long time studied question as it has been considered in different areas of mathematics, physics, chemistry and computer science, since its introduction by Kirchhoff in 1847 [7]. This graph invariant is a parameter that characterizes the reliability of a network [3, 11] and can be related to its optimal synchronization [14] and the study of random walks [8]. It is also of interest in theoretical chemistry, see for example [1]. The number of spanning trees of a graph can be computed, as shown in many basic texts on graph theory, from the Kirchhoff's matrix-tree theorem and it is given by the product of all nonzero eigenvalues of the Laplacian matrix of the graph [4]. Although this result can be applied to any graph, the calculation of the number of spanning trees from the matrix theorem is analytically and computationally demanding, in particular for large networks. Not surprisingly, some recent work have been devoted to find alternative methods to produce closed-form expressions for the number of spanning trees for particular graphs such as grid graphs [9], lattices [13], the Sierpinski gasket [2], and so forth.

In this talk, we find an exact analytical expression for the number of spanning trees of the n-disc Hanoi graph. This graph comes from the well known tower of Hanoi puzzle since the graph is associated to the allowed moves in this problem [5]. There exists an abundant literature on the properties of the Hanoi graph, which includes the study of shortest paths, average distance, planarity, Hamiltonian walks, group of symmetries, to name a few problems, see for example [6, 12, 10] and references therein. Thus, our study is relevant given the importance of the graph, and because of the method used to compute the number of spanning trees which is based on the self-similarity of the Hanoi graph. Our result allows also the calculation of the spanning tree entropy of Hanoi graphs and we compare its asymptotic value with those of other graphs with the same average degree, like the honeycomb lattice [15], the 4-8-8 (bathroom tile) and 3-12-12 lattices [13]. The value for the Hanoi graph is the lowest reported for graphs with average degree 3. This reflects the fact that the number of spanning trees in Hanoi graphs, although growing exponentially, do it at a lower rate than lattices with the same average degree.

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Perturbations in Almost Distance-Regular Graphs

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1 Introduction

The aim of this paper is to put together some ideas and results from different theories to show that certain almost distance-regular graphs, the so-called h-punctually walk-regular (or h-punctually spectrum-regular) graphs, can be characterized through the cospectrality of their perturbed graphs. We consider three one-vertex perturbations, namely, vertex deletion, adding a loop at a vertex, and adding a pendant edge at a vertex. These three perturbations are extended to pairs of vertices to obtain two-vertex 'separate' perturbations. We also consider three two-vertex 'joint' perturbations, namely adding/removing an edge, amalgamating two vertices, and adding a bridging vertex. We show that for walk-regular graphs all these two-vertex operations are equivalent, in the sense that one perturbation produces cospectral graphs if and only if the others do. We also consider perturbations on a set of vertices, and their impact on almost distance-regular graphs. As a consequence, we obtain some new characterizations of distance-regular graphs, in terms of the cospectrality of their perturbed graphs.

2 Preliminaries

2.1 Graphs and their spectra

Let $G = (V, E)$ be a (connected) graph with vertex set V and edge set E. The adjacency between vertices $u, v \in V$, that is $uv \in E$, is denoted by $u \sim v$, and their distance is $\partial(u, v)$. Let $\boldsymbol{A} = (a_{uv})$ be the adjacency matrix of G, with characteristic polynomial $\phi_G(x)$, and spectrum $\operatorname{sp} G = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \dots, \lambda_d^{m_d}\}$, where the different eigenvalues of G are in decreasing order, $\lambda_0 > \lambda_1 > \cdots > \lambda_d$, and the superscripts stand for their multiplicities $m_i = m(\lambda_i)$. For $i = 0, 1, \dots, d$, let $\boldsymbol{E}_i$ be the principal idempotent of $\boldsymbol{A}$, which corresponds to the orthogonal projection onto the eigenspace $\mathcal{E}_i = \operatorname{Ker}(\lambda_i \boldsymbol{I} - \boldsymbol{A})$. In particular, if G is regular, $\boldsymbol{E}_0 = \frac{1}{n}\boldsymbol{J}$, where $\boldsymbol{J}$ stands for the all-1 matrix. As is well known, the idempotents satisfy the following properties: $\boldsymbol{E}_i\boldsymbol{E}_j = \delta_{ij}\boldsymbol{E}_i$ (with δ_{ij} being the Kronecker delta), $\boldsymbol{A}\boldsymbol{E}_i = \lambda_i\boldsymbol{E}_i$, and $q(\boldsymbol{A}) = \sum_{i=0}^{d} q(\lambda_i)\boldsymbol{E}_i$ for every rational function q that is well-defined at each eigenvalue of $\boldsymbol{A}$; see, for instance, Godsil [6]. The uv-entry $m_{uv}(\lambda_i) = (\boldsymbol{E}_i)_{uv}$ of the idempotent $\boldsymbol{E}_i$ is called the *crossed* (uv-)*local multiplicity* of λ_i. See, for example, [2]) for more details.

Note that the uv-entry $a_{uv}^{(\ell)}$ of the power matrix $\boldsymbol{A}^\ell$ is equal to the number of walks of length ℓ between vertices u, v. Rowlinson [11] showed that a graph G is distance-regular if and only if this number of walks only depends on $\ell = 0, 1, \ldots, d$ and the distance $\partial(u, v)$ between u and v. Similarly, G is distance-regular if and only if its local crossed multiplicities $m_{uv}(\lambda_i)$ only depend on λ_i and $\partial(u, v)$; see [3]. Inspired by these characterizations, the authors [1] introduced the following concepts as different approaches to 'almost distance-regularity'. We say that a graph G with diameter D and $d + 1$ distinct eigenvalues is *h-punctually walk-regular*, for a given $h \le D$, if for every $\ell \ge 0$ the number of walks of length ℓ between a pair of vertices u, v at distance $\partial(u, v) = h$ does not depend on u, v. Similarly, we say that G is *h-punctually spectrum-regular*, for a given $h \le D$ if for all $i \le d$, the crossed uv-local multiplicities of λ_i are the same for all pairs of vertices u, v at distance $\partial(u, v) = h$. In this case, we write $m_{uv}(\lambda_i) = m_{hi}$. The concepts of h-punctual walk-regularity and h-punctual spectrum-regularity are equivalent. For $h = 0$, the concepts are equivalent to walk-regularity (a concept introduced by Godsil and McKay in [7]) and spectrum-regularity (see [4]), respectively.

2.2 Graph perturbations

As mentioned above, we consider three basic graph perturbations which involve a given vertex $u \in V$:
P1. $G - u$ is the graph obtained from G by removing u and all the edges incident to it.
P2. $G + uu$ is the (pseudo)graph obtained from G by adding a loop at u. (In this case the graph obtained has adjacency matrix as expected, with its uu-entry equal to 1.)
P3. $G + u\bar{u}$ is the graph obtained from G by adding a pendant edge at u (thus creating a new vertex $\bar{u}$).

Two vertices u, v satisfying $\mathrm{sp}(G-u) = \mathrm{sp}(G-v)$ were called *cospectral* by Herndon and Ellzey [8]. We say that a graph is 0-*punctually cospectral* when all its vertices are cospectral; a concept that we generalize below. It is well-known that a graph is 0-punctually cospectral if and only if it is walk-regular; see Proposition 3.1, where we also relate this to the perturbations **P2** and **P3**. In fact, Proposition 3.1 implies that cospectral vertices u, v can be equivalently defined by requiring that $\mathrm{sp}(G + uu) = \mathrm{sp}(G + vv)$ or $\mathrm{sp}(G + u\bar{u}) = \mathrm{sp}(G + v\bar{v})$.

Given a vertex subset $U \subset V$, we can also consider the graphs obtained by applying any of the above perturbations to every vertex of U, with natural notation $G - U$, $G + UU$ and $G + U\bar{U}$. In particular, when $U = \{u, v\}$, we also write $G - u - v$, $G + uu + vv$ and $G + u\bar{u} + v\bar{v}$.

Building on the concept of cospectral vertices, Schwenk [12] considered the analogue for sets: Two vertex subsets $U, U' \subset V$ are *removal-cospectral* if there exists a one-to-one mapping $U \to U'$ such that, for every $W \subset U$, the graphs $G - W$ and $G - W'$ are cospectral. A main result of his paper was the following necessary condition for two sets being removal-cospectral:

Theorem 2.1 *[12] If U, U' are removal-cospectral sets, then $a_{uv}^{(\ell)} = a_{u'v'}^{(\ell)}$ for all pairs of vertices $u, v \in U$ and all $\ell \geq 0$.*

Godsil [5] proved that two vertex subsets U, U' are removal-cospectral if and only if for every subset $W \subset U$ with at most two vertices, the subsets W, W' are removal-cospectral (for both an alternative proof and a geometric interpretation of this result, see Rowlinson [10]).

As a consequence of Theorem 2.1, notice that for $\{u, v\}$ and $\{u', v'\}$ to be removal-cospectral we need that $\partial(u, v) = \partial(u', v')$. Otherwise, if $r = \partial(u, v) < \partial(u', v')$, say, we would have $a_{uv}^{(r)} > 0$ whereas $a_{u'v'}^{(r)} = 0$. Inspired by this property, we say that two vertex subsets are *isometric* when there exists a one-to-one mapping $U \to U'$ such that, for every pair $u, v \in U$, we have $\partial(u, v) = \partial(u', v')$. So, if two sets are removal-cospectral then they are also isometric. In the last section, we show that the converse is also true for distance-regular graphs.

For example, in the Petersen graph all cocliques (that is, independent sets) of size 3 are removal-cospectral (see Fig. 1). By adding edges to the cocliques also gives cospectral but non-isomorphic graphs since, as was proved by Schwenk [12], if U and U' are removal-cospectral sets, then any graph may be attached to all the points of U and to the points of U' with the two graphs so formed being cospectral.

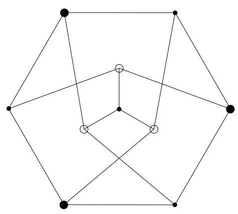

Figure 1: Petersen graph with 3-cocliques

In our framework of almost distance-regular graphs, the case when the two vertices of W are at a given distance proves to be specially relevant, and leads us to the following definition: A graph G with diameter D is *h-punctually cospectral*, for a given $h \leq D$, when, for *all* pairs of vertices u, v and w, z, both at distance $\partial(u, v) = \partial(w, z) = h$, we have $\mathrm{sp}(G - u - v) = \mathrm{sp}(G - w - z)$. Again, we show later (in Lemma 4.1) that this concept can also be defined by using the other graph perturbations considered here. Notice that, since there are no restrictions on either pair of vertices, except for their distance, this is equivalent to the sets $W = \{u, v\}$ and $W' = \{u', v'\}$, with both mappings $u' = w$, $v' = z$ and $u' = z$, $v' = w$, being removal-cospectral.

Then, using our terminology, Schwenk's theorem implies the following corollary:

Corollary 2.2 *If a graph G is j-punctually cospectral for $j = 0, h$, then it is j-punctually walk-regular for $j = 0, h$.*

Answering a question of Schwenk [12], Rowlinson [10] proved the following characterization of removal-cospectral sets, which we give in terms of the local crossed multiplicities:

Theorem 2.3 *[10] The vertex (non-empty) subsets U, U' are removal-cospectral if and only if $m_{uv}(\lambda_i) = m_{u'v'}(\lambda_i)$ for all $u, v \in U$ and $i = 0, 1, \ldots, d$.*

In our context we have the following consequence:

Corollary 2.4 *A graph G is j-punctually cospectral for $j = 0, h$ if and only if it is j-punctually spectrum-regular for $j = 0, h$.*

We remind the reader that the concepts of h-punctually walk-regularity and h-punctually spectrum-regularity are equivalent, so Corollary 2.4 implies Corollary 2.2.

3 Walk-regular graphs

Our main results were inspired by the following characterizations of walk-regular graphs:

Proposition 3.1 *For all vertices u, v, we have the following equivalences:*
G is walk-regular $\Leftrightarrow$ G is spectrum-regular $\Leftrightarrow$ $\mathrm{sp}(G-u) = \mathrm{sp}(G-v)$
$\Leftrightarrow \mathrm{sp}(G+uu) = \mathrm{sp}(G+vv) \Leftrightarrow \mathrm{sp}(G+u\bar{u}) = \mathrm{sp}(G+v\bar{v})$.

Thus, a graph G is (0-punctually) walk-regular or (0-punctually) spectrum-regular if and only it is 0-punctually cospectral, a concept which, as was claimed, can be defined through any of the considered graph perturbations. In the next section, we generalize this result.

4 h-Punctually walk-regular graphs

4.1 Separate perturbations

Let us first prove the following lemma concerning perturbations **P1-P3** for pairs of vertices in walk-regular graphs:

Lemma 4.1 *For all pairs of vertices u, v and w, z of a walk-regular graph G, we have the following equivalences:*
$\mathrm{sp}(G - u - v) = \mathrm{sp}(G - w - z) \Leftrightarrow \mathrm{sp}(G + uu + vv) = \mathrm{sp}(G + ww + zz)$
$\Leftrightarrow \mathrm{sp}(G + u\bar{u} + v\bar{v}) = \mathrm{sp}(G + w\bar{w} + z\bar{z})$.

Notice that, by this result and Proposition 3.1, each of the above conditions (a)-(c) is equivalent to the sets $\{u, v\}$ and $\{w, z\}$ being removal-cospectral. Moreover, as mentioned before, this allows us to define h-punctually cospectrality by requiring that every pair of vertices at distance h satisfies one of these conditions.

In turn, this leads to the following characterization of h-punctually walk-regular graphs. It is, in a sense, a restatement of Corollary 2.4.

Theorem 4.2 *For a walk-regular graph G with diameter D and a given integer $h \leq D$, we have the following equivalences:*
G is h-punctually walk-regular $\Leftrightarrow$ G is h-punctually spectrum-regular
$\Leftrightarrow$ G is h-punctually cospectral.

4.2 Joint perturbations

We now consider the following perturbations involving two given vertices u, v:
P4. $G \pm uv$ is the graph obtained from G by flipping the (non-)edge uv. (That is, changing the edge uv into a non-edge or vice versa.)
P5. G_{u+v} is the (pseudo)graph obtained from G by amalgamating the vertices u and v (if $u \sim v$ then the edge uv becomes a loop; if u and v have common neighbors, then multiple edges arise; the 'new' vertex is denoted by $u + v$).
P6. $G + u\bar{u}v$ is the graph obtained from G by adding the 2-path $u\bar{u}v$ (thus creating a new so-called *bridging vertex* $\bar{u}$).

In the case that the graphs $G + u\bar{u}v$ and $G + w\bar{w}z$ are cospectral, the pairs (u, v) and (w, z) are called *isospectral*; see Lowe and Soto [9]. The following result states that for walk-regular graphs, isospectral pairs can also be defined by requiring cospectrality of the graphs obtained from perturbations **P4-P5**.

Proposition 4.3 *Let u, v and w, z be pairs of vertices of a walk-regular graph G such that $u \sim v$ if and only if $w \sim z$. Then, we have the following equivalences:*
$\mathrm{sp}(G \pm uv) = \mathrm{sp}(G \pm wz) \Leftrightarrow \mathrm{sp}\, G_{u+v} = \mathrm{sp}\, G_{w+z} \Leftrightarrow \mathrm{sp}(G + u\bar{u}v) = \mathrm{sp}(G + w\bar{w}z)$.

We remind the reader that the condition $m_{uv}(\lambda_i) = m_{wz}(\lambda_i)$ for all $i = 0, 1, \ldots, d$ implies that u and v are at the same distance as w and z. Inspired by this and the above result, we say that a graph G with diameter D is *h-punctually isospectral*, for a given $h \leq D$, when every pair of vertices at distance h satisfies one of the conditions in

Proposition 4.3. As a corollary of its proof, we then obtain the following characterization of h-punctually walk-regular (or h-punctually spectrum-regular) graphs.

Corollary 4.4 *For a walk-regular graph G with diameter D and a given integer $h \leq D$, we have the following equivalences:*
G is h-punctually walk-regular $\Leftrightarrow$ G is h-punctually spectrum-regular
$\Leftrightarrow$ G is h-punctually isospectral.

4.3 Multiple perturbations

For the sake of simplicity, we have only considered perturbations in a single graph G so far. However, we also could use the above perturbations in cospectral graphs G and G' to get new cospectral graphs (as is well known from the literature). The conditions for this to work are similar as before: the crossed local multiplicities $m_{uv}(\lambda_i)$ (in G) and $m'_{wz}(\lambda_i)$ (in G') should be the same for all $i = 0, 1, \ldots, d$ (or alternatively: the number of walks $a_{uv}^{(\ell)}$ (in G) and $a'^{(\ell)}_{wz}$ (in G') should be the same for all ℓ).

For the next step — multiple perturbations — it is hard to avoid working with different (but cospectral) graphs. We next consider removal-cospectral sets U, U' belonging to cospectral (but not necessarily isomorphic) graphs G, G' (i.e., there exists a one-to-one mapping $U \to U'$ such that, for every $W \subset U$, the graphs $G - W$ and $G' - W'$ are cospectral), as is usually done in the literature. The following proposition shows that all perturbations **P1-P6** leave the property of two sets being removal-cospectral invariant, and gives new insight into some of the previous implications.

Proposition 4.5 *Let U and U' be removal-cospectral sets in cospectral graphs G and G', and let $u, v \in U$ with corresponding vertices $u', v' \in U'$. Let $\widetilde{U}, \widetilde{U}'$ be the sets obtained from U, U' after perturbing vertices u and u' according to one of the perturbations* **P1-P3***, or perturbing pairs of vertices u, v and u', v' through one of the perturbations* **P4-P6***, where possible new vertices $u + v$, $\bar{u}$, $\bar{u}'$ are included in $\widetilde{U}, \widetilde{U}'$. Let $\widetilde{G}$ and $\widetilde{G}'$ be the resulting perturbed graphs. Then, the sets $\widetilde{U}, \widetilde{U}'$ are removal-cospectral in $\widetilde{G}$ and $\widetilde{G}'$.*

As a consequence, notice that the different one-vertex and two-vertex perturbations can be repeated over and over again to obtain different cospectral graphs $\widetilde{G}$ and $\widetilde{G}'$. In other words, from two removal-cospectral sets U, U', one can, for example, amalgamate several vertices, or combine amalgamation with other operations such an edge removal/addition (hence also contract an edge), adding pendant edges, etc., to obtain new removal-cospectral sets $\widetilde{U}, \widetilde{U}'$ in the corresponding cospectral graphs $\widetilde{G}, \widetilde{G}'$. This suggests the following definition: Two vertex subsets U, U' of cospectral graphs G, G' are called *perturb-cospectral* if for all subsets $S \subset U$ and $S' \subset U'$, the perturbed graphs $\widetilde{G}$ and $\widetilde{G}'$, obtained by applying **P1-P6** to corresponding vertices of U and U', are cospectral.

5 Distance-regular graphs

In this section we use the above results to obtain some new characterizations of distance-regular graphs.

In [1], the authors considered also the following concepts: A graph G is *m-walk-regular* (respectively *m-spectrum-regular*) when it is i-punctually walk-regular (respectively i-punctually spectrum-regular) for every $i \leq m$. Similarly, we say that G is *m-cospectral* (respectively, *m-isospectral*) when it is i-punctually cospectral (respectively, *i-punctually isospectral*) for every $i \leq m$. Using these definitions, Theorem 4.2 and Corollary 4.4 have the following direct consequence:

Corollary 5.1 *For a walk-regular graph G with diameter D and a given integer $m \leq D$, we have the following equivalences:*
G is m-walk-regular $\Leftrightarrow$ G is m-spectrum-regular $\Leftrightarrow$ G is m-cospectral $\Leftrightarrow$ G is m-isospectral.

Moreover, as mentioned in Section 2.1, Rowlinson [11] proved that a graph G is distance-regular if and only if it is D-walk-regular. Hence, we get the following characterization:

Theorem 5.2 *Let G be a graph with diameter D. Then, we have the following equivalences:*
G is distance-regular $\Leftrightarrow$ G is D-cospectral $\Leftrightarrow$ G is D-isospectral.

In fact, notice that we also proved the following result:

Theorem 5.3 *A graph $G = (V, E)$ is distance-regular if and only if every two isometric subsets $U, U' \subset V$ are perturb-cospectral.*

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Charles Delorme

Symmetric graphs and $GL(2, R)$

I intend to describe some constructions of symmetric graphs, some are large and some are large among graphs subect to other conditions, like persistance of the diameter after romoval of a vertex or an edge.

The tools will be of course Cayley graphs and quotients of Cayley graphs by subgroups.

The groups $GL(2, R)$ of 2×2 matrices on a ring R are mainly used, because of the geometrical interpretation of some constructions.

Some famous graphs and families of graphs can be obtained in this way. Is it necessary to indicate that Petersen graph, Tutte-Coxeter graph, Heawood graph, Biggs-Smith graph (102 vertices) invite themselves in the list?

Indeed the group (or rather its quotient $PGL(2, R)$) operates on the projective line on R by homographies.

The group with the ring $\mathbb{R}$ may be used to modelize rotations, and in particular rotations of platonic polyhedra, suggesting similar objects for finite fields

Ramanujan graphs (not always symmetric) can be built within a similar frame.

Some references:

- Ramanujam graph and the reason why they tend to have a large girth (and therefor a moderate diameter) is explained in

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- The construction of graphs with involutive homographies on the projective line was inspired by

  Biggs, Hoare. The sextet construction for cubic graphs. Combinatorica 3 p.153-165 (1983)

- Characterising vertex-transitive graphs is a result in

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A Survey of Generalized Connectivity and its Applications in Security

Yvo DESMEDT

ABSTRACT

In the first part of the talk, we survey the use of AND/OR graphs to model critical infrastructures. It was shown that the equivalent problem of connectivity for AND/OR graphs is Co-NP-complete. When adding to each edge a capacity, other critical aspects of infrastructures can be modeled, as briefly explained.

In the second part of the talk we focus on graphs and directed graphs. In a k-connected graph, the destruction of k-1 nodes will continue to guarantee a path between any two nodes in the graph. In practice the idea that faults can be modeled using a threshold is misleading. An attack today can be replicated using computer viruses, worms and hybrids. So, instead of being able to disrupt k nodes, the adversary might be able to choose k platforms and shut down all nodes running one of these k platforms. Labeling the vertices allows to model this. Indeed, the label indicates the operating system a computer or router (i.e., a node) uses. Guaranteeing that the remaining graph is still connected is Co-NP-complete. A similar problem consists of labeling the edges. We link this problem to work by Euler and explain the relevance to designing computer networks in earthquake zones.

In the third part of the talk we explain the role connectivity plays in cryptography. The issue at stake is how a sender and a receiver can privately and reliable communicate even though they have not exchanged a key and they do not trust RSA (etc.)

Although connectivity is an old problem, above generalizations have only been
studied during the last 15 years. Several research groups on cryptography in the world have contributed to this line of work.

Edge-Distance-Regularity *

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1 Introduction

Since its introduction by Biggs in the 70's (see [1]), the theory of distance-regular graphs has been widely developed in the last decades. Its importance is highlighted in the preface's comment of the comprehensive textbook of Brower, Cohen and Neumaier [2]: "Most finite objects bearing 'enough regularity' are closely related to certain distance-regular graphs."

When we look at the distance partition of the graph from each of its edges instead of its vertices, we arrive, in a natural way, to the concept of edge-distance-regularity [6]. More precisely, a graph Γ with adjacency matrix $\boldsymbol{A}$ is edge-distance-regular when it is distance-regular around each of its edges and with the same intersection numbers for any edge taken as a root. In this work we study this concept, give some of its properties, such as the regularity of Γ, and derive some characterizations. In particular, it is shown that a graph is edge-distance-regular if and only if its k-incidence matrix is a polynomial of degree k in $\boldsymbol{A}$ multiplied by the (standard) incidence matrix. Also, the analogue of the spectral excess theorem for distance-regular graphs is proved, so giving a quasi-spectral characterization of edge-distance-regularity. Finally, it is shown that every nonbipartite graph which is both distance-regular and edge-distance-regular is a generalized odd graph.

Let Γ be a connected graph with adjacency matrix $\boldsymbol{A}$. Its spectrum is denoted by $\operatorname{sp}\Gamma = \{\lambda_0^{m(\lambda_0)}, \lambda_1^{m(\lambda_1)}, \ldots, \lambda_d^{m(\lambda_d)}\}$, where the eigenvalues are listed in decreasing order and $m(\lambda_l)$ is the multiplicity of λ_l as an eigenvalue of $\boldsymbol{A}$. Let $\operatorname{ev}\Gamma$ be the set of different eigenvalues of Γ. The principal idempotents of $\boldsymbol{A}$ are denoted by $\boldsymbol{E}_l$, $l = 0, 1, \ldots, d$. The Perron-Frobenius Theorem ensures that $m(\lambda_0) = 1$ and guaranties the existence of a positive

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eigenvector $\boldsymbol{\nu} \in \ker(\boldsymbol{A} - \lambda_0 \boldsymbol{I})$ with minimum component equal to one. Given a nonempty set C of vertices of Γ, we consider the map $\boldsymbol{\rho} : \mathcal{P}(V) \to \mathbb{R}^n$ defined by $\boldsymbol{\rho}\emptyset = \vec{0}$ and $\boldsymbol{\rho}C = \sum_{i \in C} \nu_i \boldsymbol{e}_i$ for $C \neq \emptyset$ and denote by $\boldsymbol{e}_C$ the normalized of the vector $\boldsymbol{\rho}C$. If $\boldsymbol{e}_C = \boldsymbol{z}_C(\lambda_0) + \boldsymbol{z}_C(\lambda_1) + \cdots + \boldsymbol{z}_C(\lambda_d)$ is the spectral decomposition of $\boldsymbol{e}_C$, that is $z_C(\lambda_l) = \boldsymbol{E}_l \boldsymbol{e}_C$, the *C-multiplicity* of the eigenvalue λ_l is defined by $m_C(\lambda_l) = \|\boldsymbol{z}_C(\lambda_l)\|^2$. We denote by $\operatorname{ev}_C \Gamma = \{\mu_0, \mu_1, \ldots, \mu_{d_C}\}$ the set of different eigenvalues with non-zero C-multiplicity and write $\operatorname{sp}_C \Gamma = \{\mu_0^{m_C(\mu_0)}, \mu_1^{m_C(\mu_1)}, \ldots, \mu_{d_C}^{m_C(\mu_{d_C})}\}$ for the *C-spectrum* of Γ. As an analogous to the relation between the diameter of a graph and its number of different eigenvalues, the eccentricity of C is bounded by $\varepsilon_C \leq d_C$, and when equality is attained we say that C is an *extremal* set. If C is a single vertex u, the u-local multiplicities coincide with the diagonal entries of the idempotents, $m_i(\lambda_l) = (\boldsymbol{E}_l)_{uu}$. By analogy, for every pair of vertices $u, v \in V$, the uv-crossed multiplicity of λ is $m_{uv}(\lambda_l) = (\boldsymbol{E}_l)_{uv}$.

Let $\mathcal{M} = \{\lambda_0 > \lambda_1 > \ldots > \lambda_d\}$ be a mesh of real numbers and $g : \mathcal{M} \to \mathbb{R}$ a weight function defined on it. In $\mathbb{R}[x]/\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by the polynomial $Z(x) = \prod_{l=0}^{d}(x - \lambda_l)$, we define the inner product associated to $(\mathcal{M}, g)$ by

$$\langle p, q \rangle = \sum_{l=0}^{d} g(\lambda_l) p(\mu_l) q(\mu_l).$$

The canonical orthogonal system associated to $(\mathcal{M}, g)$ is the unique family of polynomials $\{p_k\}_{0 \leq k \leq d}$ with $\deg p_k = k$ and $\|p_k\|^2 = p_k(\lambda_0)$. See [3] for a comprehensive study of this family. The *C-local predistance polynomials* $\{p_k^C\}_{0 \leq k \leq d_C}$ are the canonical orthogonal system associated to the mesh $\operatorname{ev}_C \Gamma$, with weight function $m_C : \operatorname{ev}_C \Gamma \to \mathbb{R}$ given by the C-multiplicities. Similarly, the predistance polynomials $\{p_k\}_{0 \leq k \leq d}$ are the canonical orthogonal system associated to $\operatorname{ev} \Gamma$ and weight function given by $g(\lambda_l) = m(\lambda_l)/n$.

2 Pseudo-regular partitions and edge-distance-regularity

Given a set C of vertices of a simple connected graph $\Gamma = (V, E)$ with eccentricity ε_C, consider the partition of V given by the distance to C: $V = C_0 \cup C_1 \cup \cdots \cup C_{\varepsilon_C}$, where $C_k = \{i \in V \mid \partial(i, C) = k\}$. We say that Γ is *C-local pseudo-distance-regular* whenever this partition of the vertex set is pseudo-regular, that is, when the numbers

$$c_k(i) = \frac{1}{\nu_i} \sum_{j \in \Gamma(i) \cap C_{k-1}} \nu_j, \quad a_k(i) = \frac{1}{\nu_i} \sum_{j \in \Gamma(i) \cap C_k} \nu_j, \quad b_k(i) = \frac{1}{\nu_i} \sum_{j \in \Gamma(i) \cap C_{k+1}} \nu_j,$$

with ν_i being the i-th component of the unique positive eigenvector of the adjacency matrix of Γ with minimum component equal to one, $\boldsymbol{\nu}$, do not depend on the chosen vertex $i \in C_k$, but only on the value of k. If this is the case, we denote them simply by c_k, a_k and b_k and call them the *pseudo-intersection numbers*. When the considered graph Γ is regular, these parameters coincide with the usual intersection numbers and, in this case, Γ is C-local pseudo-distance-regular if and only if C is a completely regular code.

Notice that when a graph is $\{i\}$-local pseudo-distance regular for every vertex i and with the same intersection numbers, it is distance-regular. By considering edges as sets of two vertices, we can also see the graph from a global point of view. In this way, a connected graph Γ is edge-distance-regular when it is e-local pseudo-distance-regular for every edge $e \in E$, with intersection numbers not depending on $e \in E$.

Several quasi-spectral characterizations are known for local pseudo-distance-regularity, most of them obtained through the predistance polynomials [5, 7]. In this paper we develop the study of edge-distance-regularity and prove similar results to those known for (vertex) distance-regularity.

3 Edge-spectrum regularity

Formally, we do not distinguish between an edge $e \in E$ joining vertices u, v and the set $\{u, v\}$. Thus, we denote the (*local*) *e-multiplicities* of Γ as $m_e(\lambda_i) = \|\boldsymbol{E}_i \boldsymbol{e}_e\|^2$, $i = 0, 1, \ldots, d$, where

$$\boldsymbol{e}_e = \frac{\boldsymbol{\rho}_e}{\|\boldsymbol{\rho}_e\|} = \frac{\nu_u \boldsymbol{e}_u + \nu_v \boldsymbol{e}_v}{\sqrt{\nu_u^2 + \nu_v^2}}.$$

From this, note that the relationship between the e-multiplicity and the local and crossed multiplicities of u and v is

$$m_e(\lambda_i) = \frac{1}{\nu_u^2 + \nu_v^2}(\nu_u^2 m_u(\lambda_i) + 2\nu_u \nu_v m_{uv}(\lambda_i) + \nu_v^2 m_v(\lambda_i)).$$

If $|\mathrm{ev}_e\, \Gamma| = d_e + 1$, the eccentricity of e, seen as a set of two vertices, satisfies $\varepsilon_e \leq d_e$. We define the *edge-diameter* of Γ by $\tilde{D} = \max_{e \in E} \varepsilon_e$. Notice that $\tilde{D}$ coincides with the diameter of the line graph $L\Gamma$ of Γ. Consequently, if Γ have diameter D we have $D - 1 \leq \tilde{D} \leq D$ and, if Γ is bipartite, $\tilde{D} = D - 1$.

Lemma 3.1 *The e-multiplicities of a connected graph $\Gamma = (V, E)$ with spectrum* $\mathrm{sp}\,\Gamma$ *satisfy the following properties:*

(*a*) $\displaystyle\sum_{i=0}^{d} m_e(\lambda_i) = 1$ *for every $e \in E$.*

(*b*) *If Γ is regular, then* $\displaystyle\sum_{e \in E} m_e(\lambda_i) = \frac{\lambda_0 + \lambda_i}{2} m(\lambda_i)$ *for every* $\lambda_i \in \mathrm{ev}\,\Gamma$.

For every eigenvalue $\lambda_i \in \mathrm{ev}\,\Gamma$, the *mean vertex-multiplicity* and *mean edge-multiplicity* are, respectively,

$$g(\lambda_i) = \frac{1}{|V|} \sum_{u \in V} m_u(\lambda_i) = \frac{m(\lambda_i)}{|V|}, \qquad \tilde{g}(\lambda_i) = \frac{1}{|E|} \sum_{e \in E} m_e(\lambda_i).$$

Moreover, if Γ is regular, Lemma 3.1 gives:

$$\tilde{g}(\lambda_i) = \frac{1}{|E|}\frac{\lambda_0 + \lambda_i}{2}m(\lambda_i) = \frac{1}{\lambda_0|V|}(\lambda_0 + \lambda_i)m(\lambda_i) = \left(1 + \frac{\lambda_i}{\lambda_0}\right)g(\lambda_i).$$

Inspired by the concept of (vertex) spectrum-regularity, we say that Γ is *edge-spectrum-regular* if, for every $\lambda_i \in \operatorname{ev}\Gamma$, the edge-multiplicity $m_e(\lambda_i)$ does not depend on $e \in E$. Whereas spectrum-regularity implies regularity, in the case of edge-spectrum-regularity we have the following result.

Proposition 3.2 *Let Γ be a connected edge-spectrum-regular graph. Then, Γ is either regular or bipartite biregular.*

We say that a graph Γ is *bispectrum-regular* when it is both spectrum-regular and edge-spectrum-regular. A graph is m-walk-regular if the number of walks of length k joining two vertices depends only on the distance between them, provided that this distance is at most m. Notice that a distance-regular graph with diameter D is D-walk-regular and 0-walk-regularity corresponds to walk-regularity.

Proposition 3.3 *Γ is bispectrum-regular if and only if it is 1-walk-regular.*

4 Edge-distance-regularity

Given a connected graph $\Gamma = (V, E)$ and an edge $e \in E$, consider the partition of V induced by the distance from e, that is $V = e_0 \cup e_1 \cup \cdots \cup e_{\varepsilon_e}$, where $e_k = \Gamma_k(e)$. We say that Γ is *e-local pseudo-distance-regular* if this partition is pseudo-regular. One of the advantages of considering edges is that we can see the graph from a global point of view, that is, from every edge, in the same way as we get distance-regularity by seeing the graph from every vertex.

Definition 4.1 *A graph Γ is edge-distance-regular when it is e-local pseudo-distance-regular for every edge $e \in E$, with intersection numbers not depending on $e \in E$.*

Proposition 4.2 *Let Γ be an edge-distance-regular graph with diameter D and $d + 1$ distinct eigenvalues. Then, Γ is regular and*

(a) *Γ has spectrally maximum diameter ($D = d$) and its edge-diameter satisfies $\tilde{D} = D$ if Γ is nonbipartite and $\tilde{D} = D - 1$ otherwise.*

(b) *Γ is edge-spectrum regular and, for every $e \in E$, the e-spectrum satisfies:*

  (b1) *If Γ is nonbipartite, $\operatorname{ev}_e \Gamma = \operatorname{ev}\Gamma$ and $m_e(\lambda_i) = \left(1 + \frac{\lambda_i}{\lambda_0}\right)\frac{m(\lambda_i)}{|V|}$, $\lambda_i \in \operatorname{ev}\Gamma$.*

(*b2*) *If* Γ *is bipartite,* $\mathrm{ev}_e \Gamma = \mathrm{ev}\,\Gamma \setminus \{-\lambda_0\}$ *and* $m_e(\lambda_i) = \left(1 + \frac{\lambda_i}{\lambda_0}\right) \frac{m(\lambda_i)}{|V|}$, $\lambda_i \in \mathrm{ev}\,\Gamma \setminus \{-\lambda_0\}$.

Definition 4.3 *The k-incidence matrix of $\Gamma = (V, E)$ is the $(|V| \times |E|)$-matrix $\boldsymbol{B}_k = (b_{ue})$ with entries $b_{ue} = 1$ if $\partial(u, e) = k$, and $b_{ue} = 0$ otherwise.*

Theorem 4.4 *A regular graph Γ with edge-diameter $\tilde{D}$ is edge-distance-regular if and only if, for every $k = 0, 1, \ldots, \tilde{D}$, there exists a polynomial $\tilde{p}_k$ of degree k such that $\tilde{p}_k(\boldsymbol{A})\boldsymbol{B}_0 = \boldsymbol{B}_k$.*

Godsil and Shawe-Taylor [8] defined a distance-regularized graph as that being distance-regular around each of its vertices (these graphs are a common generalization of distance-regular graphs and generalised polygons). They showed that distance-regularized graphs are either distance-regular or distance-biregular. Inspired by this, we introduce the following concept.

Definition 4.5 *A regular graph Γ is said to be edge-distance-regularized when it is edge-distance-regular around each of its edges.*

Let $\mathrm{ev}_E \Gamma = \bigcup_{e \in E} \mathrm{ev}_e \Gamma$ and denote by $\mathrm{ev}^{\star}_E \Gamma = \mathrm{ev}_E \Gamma \setminus \{\lambda_0\}$ and $\tilde{d} = |\mathrm{ev}^{\star}_E \Gamma|$. If Γ is edge-distance-regular, Proposition 4.2 establishes that $\mathrm{ev}_E \Gamma = \mathrm{ev}\,\Gamma$ if Γ is nonbipartite, and $\mathrm{ev}_E \Gamma = \mathrm{ev}\,\Gamma \setminus \{\lambda_0\}$ otherwise. Consider the canonical orthogonal system $\{\tilde{p}_k\}_{0 \le k \le \tilde{d}}$ associated to $(\mathrm{ev}_E \Gamma, \tilde{g})$, and their sum polynomials $\{\tilde{q}_k\}_{0 \le k \le \tilde{d}}$ defined by $\tilde{q}_k = \tilde{p}_0 + \tilde{p}_1 + \cdots + \tilde{p}_k$.

Theorem 4.6 *Let $\Gamma = (V, E)$ be a regular graph with $\tilde{d} = |\mathrm{ev}_E \Gamma|$. Let $H_{\tilde{d}-1}$ be the harmonic mean of the numbers $|N_{\tilde{d}-1}(e)|$ for $e \in E$. Then, Γ is edge-distance-regularized if and only if $H_{\tilde{d}-1} = 2\tilde{q}_{\tilde{d}-1}(\lambda_0)$.*

Using that the harmonic mean is always smaller than or equal to the arithmetic mean and the relation between the sum polynomials and the predistance polynomials we get:

Corollary 4.7 *Let $\Gamma = (V, E)$ be a regular graph with $\tilde{d} = |\mathrm{ev}_E \Gamma|$. Let $M_{\tilde{d}}$ be the (arithmetic) mean of the numbers $|e_{\tilde{d}}|$ for $e \in E$. Then, Γ is edge-distance-regularized if and only if $M_{\tilde{d}} = 2\tilde{p}_{\tilde{d}}(\lambda_0)$.*

As a consequence we have the following theorem, which can be seen as an analogue for the Spectral Excess Theorem for (vertex) distance-reguarity [7, 9].

Theorem 4.8 *A regular graph $\Gamma = (V, E)$ with $\tilde{d} = |\mathrm{ev}_E \Gamma|$ is edge-distance-regular if and only if, for every edge $e \in E$, $|e_{\tilde{d}}| = 2\tilde{p}_{\tilde{d}}(\lambda_0)$.*

We remark that, as proved in [3], we can specify the value of $\tilde{p}_{\tilde{d}}$ in terms of the edge spectrum. In what follows, $\pi_0 = \prod_{j=1}^{d}(\lambda_0 - \lambda_j)$, $\widehat{\pi}_0 = \prod_{j=1}^{d}(\lambda_0 + \lambda_j)$ and $\overline{\pi}_i = (\lambda_i + \lambda_0)\prod_{j=0, j\neq i}^{\tilde{d}} |\lambda_i - \lambda_j|$, $0 \leq i \leq d$, are moment-like parameters computed from the spectrum.

Theorem 4.9 *Let $\Gamma = (V, E)$ be a regular graph with $d+1$ distinct eigenvalues, and spectrally maximum edge-diameter $\tilde{D} = \tilde{d}$. Then, Γ is edge-distance-regular if and only if, for every edge $e \in E$,*

$$|e_{\tilde{D}}| = \frac{4|E|}{\overline{\pi}_0^2}\left(\sum_{i=0}^{d} \frac{\lambda_0 + \lambda_i}{m(\lambda_i)\overline{\pi}_i^2}\right)^{-1}.$$

Proposition 4.10 *Let Γ be a λ_0-regular graph with edge-diameter $\tilde{D} = |\,\mathrm{ev}^\star\,\Gamma| = d$. Assume that, for every vertex $u \in V$ and every edge $e \in E$,*

$$\frac{|e_d|}{|u_d|} = \frac{\widehat{\pi}_0}{\pi_0}\frac{|V|}{(-1)^d p_d(-\lambda_0)},$$

where p_d is the d-th predistance polynomial of Γ. Then, Γ is edge-distance-regular if and only if it is distance-regular.

A distance-regular graph Γ with diameter D and odd-girth (that is, the shortest cycle of odd length) $2D+1$ is called a *generalized odd graph*, also known as an *almost-bipartite distance-regular graph* or a *regular thin near* $(2D+1)$*-gon*. The name is due to the fact that the odd graphs O_k fulfil such conditions [1]. In this case, the intersection parameters of Γ satisfy $a_0 = a_1 = \cdots = a_{d-1} = 0$ and $a_d \neq 0$. Van Dam and Haemers [10] showed that any connected regular graph with $d+1$ distinct eigenvalues and odd-girth $2d+1$ is a generalized odd graph. In [4], it was shown, through an algebraic approach, that the same result holds when Γ is both distance-regular and edge-distance-regular. Here, we provides a combinatorial proof.

Theorem 4.11 *Let Γ be a distance-regular graph with diameter $D = d$ and intersection array*

$$\begin{pmatrix} 0 & c_1 & \cdots & c_{d-1} & c_d \\ a_0 & a_1 & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & \cdots & b_{d-1} & 0 \end{pmatrix}.$$

Then, Γ is edge-distance-regular if and only if it is either bipartite or a generalized odd graph. In this case, when Γ is nonbipartite, the edge-intersection array is:

$$\begin{pmatrix} 0 & \tilde{c}_1 & \cdots & \tilde{c}_{d-1} & \tilde{c}_d \\ \tilde{a}_0 & \tilde{a}_1 & \cdots & \tilde{a}_{d-1} & \tilde{a}_d \\ \tilde{b}_0 & \tilde{b}_1 & \cdots & \tilde{b}_{d-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & c_1 & \cdots & c_{d-1} & 2c_d \\ c_1 & c_2 - c_1 & \cdots & c_d - c_{d-1} & a_d - c_d \\ b_1 & b_2 & \cdots & a_d & 0 \end{pmatrix},$$

and, when Γ is bipartite:

$$\begin{pmatrix} 0 & \tilde{c}_1 & \cdots & \tilde{c}_{d-2} & \tilde{c}_{d-1} \\ \tilde{a}_0 & \tilde{a}_1 & \cdots & \tilde{a}_{d-2} & \tilde{a}_{d-1} \\ \tilde{b}_0 & \tilde{b}_1 & \cdots & \tilde{b}_{d-2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & c_1 & \cdots & c_{d-2} & c_{d-1} \\ c_1 & c_2 - c_1 & \cdots & c_{d-1} - c_{d-2} & b_0 - c_{d-1} \\ b_1 & b_2 & \cdots & b_{d-1} & 0 \end{pmatrix}.$$

Proposition 4.12 *Let Γ be a nonbipartite distance-regular graph with with intersection numbers c_k, a_k, b_k, $0 \leq k \leq d$, and (pre)distance polynomials $\{p_k\}_{0\leq k\leq d}$. Then, the following statements are equivalent:*

(a) *Γ is edge-distance-regular.*

(b) *$a_0 = a_1 = \cdots = a_{d-1} = 0$ and $a_d \neq 0$.*

(c) *For every $k = 0, 1, \ldots, d$, p_k has even parity for even k and odd parity for odd k. In this case, the edge-distance polynomials $\{\tilde{p}_k\}_{0\leq k\leq d}$ are:*

$$\tilde{p}_k = p_k - p_{k-1} + p_{k-2} - \cdots + (-1)^k p_0 \qquad (0 \leq k \leq d-1),$$
$$\tilde{p}_d = \tfrac{1}{2}(p_d - p_{d-1} + p_{d-2} - \cdots + (-1)^d p_0).$$

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Power Domination: Graphs and Electrical Power Networks

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Electrical power networks must be continually monitored in order to optimize their use and to avoid blackouts. This task is usually performed by placing Phase Measurement Units (PMUs) at selected network locations. A PMU reading consists of a complex number that represents both, the magnitude and phase angle of the sine waves found in electricity. Typically, PMUs are placed at widely dispersed locations in the power system network and synchronized from the common time source of a global positioning system (GPS). However, due to the high cost of each PMU, it is important to minimize the number of PMUs used to monitor a given power network. This problem leads to an optimization problem in graph theory: given a graph modeling a power network, find the minimum number of nodes, and their locations, where PMUs must be placed in order to monitor the entire network. This problem is the referred to as the power domination problem.

Formally, the power domination problem in graphs can be formulated in the following way. Given a graph G and a set of vertices S, initially, S monitors all the vertices in it, and their neighbors. Then, we apply the following rule: if a monitored vertex has exactly one non-monitored neighbor, then the non-monitored neighbor becomes monitored. The application of this rule is iterated until it no longer detects new vertices that can be monitored. At that point, if all the vertices are monitored, then S is a power dominating set of G. Therefore, the power domination problem consists of finding a minimal power dominating set for a given graph.

The power domination decision problem has been proven to be NP-complete [T.W. Haynes, S.M. Hedetniemi, S.T. Hedetniemi, M.A. Henning,] even when reduced to certain classes of graphs, such as bipartite graphs and chordal graphs [T.W. Haynes, S.M. Hedetniemi, S.T. Hedetniemi, M.A. Henning,] or even split graphs [C.S. Liao, D.T. Lee], a subclass of chordal graphs. However, Liao and Lee presented a linear time algorithm for solving his problem on interval graphs, if the interval ordering of the graph is provided. If the interval order is not given, they provided a polylogarithmic algorithm of $O(n \log n)$ and proved that it is asymptotically optimal. Other efficient algorithms have been presented for trees [J. Kneis, D. MÄolle, S. Richter, P. Rossmanith,] and more generally, for graphs with bounded treewidth [J. Kneis, D. MÄolle, S. Richter, P. Rossmanith,]. On block graphs [G. Xu, L. Kang, E. Shan, M. Zhao], claw-free g phs M. Zhao, L. Kang, G. Chang], and generalized Petersen graphs [R. Barrera, D. Ferrero] there are upper bounds given for the power domination number. Closed formulas exist for only a few families of graphs: rectangular grids [M. Dorfling, M. Henning], hexagonal meshes [D. Ferrero, S. Varghese, A. Vijaykumar] and some families of cylinders and tori [R. Barrera, D. Ferrero].

In this talk we will present the basic concepts in relation to power domination in graphs, some known results, and some future lines of research.

Cycles in cartesian products of graphs

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Let G and H be two graphs. The *cartesian product* of G and H is the graph with vertex set $V(G) \times V(H)$ and the edge $\{(u_1, v_1), (u_2, v_2)\}$ is present in the product whenever $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or symmetricaly $v_1 = v_2$ and $u_1 u_2 \in E(G)$. It is denoted by $G \square H$.

The interest in the study of Hamiltonian properties of prisms (cartesian product of G and K2) and generally cartesian products of graphs goes back to the paper of Barnette and Rosenfeld [1]. They showed that the cartesian product of a graph G and a clique or a cycle on n vertices is Hamiltonian assuming that the maximum degree of G is less or equal to n. The necessary and sufficient condition for a graph G to have Hamiltonian prism was given by Paulraja in the paper devoted to the existence of a Hamiltonian cycle in 3-connected cubic graphs, see [8].

Since those results, many sufficient conditions fo hamiltonicity, pancyclicity and further cyclic properties in prisms or cartesian products by paths, cycles and cliques have be given, (see for example ([3], [4] [5], [6])

One of those results, due to Ozeki ([7]) is a degree conditions for prism hamiltonicity, where $\sigma_3(G)$ is the minimum degree sum of any three independant vertices :

<u>Theorem</u> 1 *Let G be a connected graph of order $n \geq 2$. If $\sigma_3(G) \geq n$, then the prism of G is hamiltonian.*

Instead of considering the existence of cycles of given length in the cartesian product, we can look for cycles containing specific vertices. For example in [2], a localisation of Ozeki's result is given, where $\sigma_3(G)$ is the minimum degree sum of any three independant vertices of S (notice that G is not necessarily connected but S has to be included in a connected subgraph of G):

<u>Theorem</u> 2 *Let G be a graph, $|V(G)| \geq 2$, $S \subseteq V(G)$ and S is 1_G-connected. If $\sigma_3(S) \geq n$ then there is a cycle in $G \square K_2$ that contains the two copies of S.*

We will survey some classical results in the field and discuss conditions for cycles through given vertices in the cartesian product of G with paths, cycles and cliques.

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Moments in Graphs

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Abstract

Let $G = (V, E)$ be a connected graph with $n = |V|$ vertices and a *weight function* ρ that assigns a nonnegative number to each of its vertices. Then, the ρ*-moment* of G at vertex u is defined as

$$M_G^\rho(u) = \sum_{v \in V} \rho(v) \operatorname{dist}(u, v)$$

where $\operatorname{dist}(\cdot, \cdot)$ stands for the distance function. Adding up all these numbers, we obtain the ρ*-moment of* G:

$$M_G^\rho = \sum_{u \in V} M_G^\rho(u) = \frac{1}{2} \sum_{u,v \in V} \operatorname{dist}(u, v)(\rho(u) + \rho(v)).$$

This parameter generalizes, or it is closely related, to some well-known graph invariants, such as the *Wiener index* $W(G)$, when $\rho(u) = 1/2$ for every $u \in V$, and the *degree distance* $D'(G)$, obtained when $\rho(u) = \delta(u)$, the degree of vertex u.

In this talk we discuss some exact formulas for computing the ρ-moment of a graph obtained by different operations, such as the hierarchical product, of graphs, in terms of the corresponding ρ-moments of their factors. As a consequence, we provide a method for obtaining nonisomorphic graphs with the same ρ-moment. In the case when the factors are trees and/or cycles, algebraic techniques (distance matrices, eigenvalues, etc) allow us to give formulas for the degree distance of their product.

Goal-minimally k-diametric graphs

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1 Introduction

This contribution is devoted to a family of extremal graphs – edge-minimal graphs with respect to diameter. So our objects of interest are graphs in which the diameter is increasing after removing of an arbitrarily chosen edge. This class of graphs is very rich and quite well explored (for example see [1], [2], [5], [6], [7], [8], [10], [15]), therefore we will focus on one of its subclass, the so-called goal-minimally k-diametric graphs (k-GMD for short). These graphs are of diameter k and such that after the removing of an arbitrary edge uv the new graph will be of diameter $k+1$, and the only pair of vertices in distance $k+1$ will be the (unordered) pair $\{u, v\}$. Graphs with this property were introduced by Kyš in 1980 (see [14]) and studied by Plesník and Gliviak (see [9], [16]).

A graph G is said to be *goal-minimal of diameter k* or *goal-minimally k-diametric* (k-GMD for short), if the diameter of G is equal to k, and for every edge $uv \in E(G)$ the inequality $d_{G-uv}(x, y) > k$ holds if and only if $\{u, v\} = \{x, y\}$. The easiest examples of such graphs are the complete graphs, which are 1-GMD.

From practical point of view the following characterization of k-GMD graphs is very useful:

Theorem 1 *Let k be a positive integer. A graph G with order at least 3 is k-GMD if and only if the following conditions hold simultaneously:*

(i) For any two non-adjacent vertices u and v in G, there exist two independent $u-v$ paths of length not exceeding k.

(ii) G has diameter k.

(iii) G has girth $k+2$.

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Kyš [14] conjectured that *for every positive integer k there exists a k-GMD graph.* The main challenge in this topic is to prove (or disprove) this conjecture. Kyš discovered such graphs only for $k = 1, 2, 3, 4, 6$, and for $k = 1, 2, 4$ he gave infinite families of k-GMD graphs. In [16] Plesník showed the first examples of k-GMD graphs for diameters $k = 5, 7, 8, 10, 12, 14$, and constructed the first infinite family of 6-GMD graphs. Recently, Štupáková in her master thesis [18] constructed an infinite family of 8-GMD and 10-GMD graphs by joining leaves of two trees with paths of length 2.

Our effort to construct new k-GMD graphs and infinite families was successful due to methods of voltage graphs and Cayley graphs. These methods together with obtained results will be described in further sections.

2 Voltage graphs

One of the methods, by which we tried to find new k-GMD graphs, was the method of voltage graphs (also known as lifts) known mainly from topological graph theory. This method is surveyed in [11] by Gross and Tucker. An important advantage of this method is that that we need not to create the entire graph (lifted graph), because many of its properties (like diameter and girth) may be directly derived from the voltages and the base graph, which is usually much smaller than the lifted graph.

By this method we were able to reconstruct the known infinite families of 1-GMD, 2-GMD and 4-GMD graphs constructed by Kyš, Gliviak and Plesník. Moreover, we designed an infinite family of 5-GMD graphs, which is the first known non-trivial infinite family for odd diameter at all.

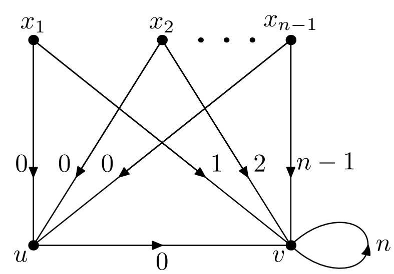

Figure 1: The voltage graph for 5-GMD graphs with voltages in $\mathbb{Z}_{2n}$.

Theorem 2 *[12] The voltage graph depicted in Figure 1 gives a 5-GMD graph for all positive integer $n \geq 2$ after lifting in the cyclic group $\mathbb{Z}_{2n}$.*

Furthermore, we showed the first examples of k-GMD graphs for diameter $k = 9$ and $k = 13$. We also designed a possible infinite family of 3-GMD graphs. This family contains infinitely many 3-GMD graphs, if there exists an infinite sequence of pairs (m, r) of integers such that there exists a relative $(m, 2, r, 1)$-difference set. Our constructions used only cyclic groups as voltage groups, so this method offers a lot of opportunities for further research. More details about our results one can find in [12].

3 Cayley graphs

The next successful construction is the well-known Cayley graph construction. For each projective special linear group $\mathrm{PSL}_2(q)$ with $q \leq 103$ we found three generators (involutions) such that the resulting cubic Cayley graph was a k-GMD graph for some integer k. This choice of the group was motivated by Conder [3] who constructed trivalent Cayley graphs in order to find small (δ, g)-graphs, i.e. small regular graphs of valency δ and girth g which are also known as near-cages.

Among our cubic Cayley graphs one can find k-GMD graphs of diameters 16, 18–24 and 26. This means that the largest integer k for which we have a k-GMD graph is 26.

Unfortunately, we did not recognize any connections between the generators of these graphs, so it would be interesting to solve the problem: *How to choose the group and the generators in order to obtain a k-GMD graph as a Cayley graph?*

4 Cages

As one could notice, by Theorem 1 follows that k-GMD graphs have fixed girth $k + 2$, and they are minimal in some sense. So we can expect that there is some relationship between them and cages – which are minimal graphs with prescribed degree and girth.

We took the list of known cages (see [4]) and checked them whether they are k-GMD or not. Except two of eighteen $(3, 9)$-cages, each known cage G satisfies the equivalence $d_{G-uv}(x, y) > k \iff \{u, v\} = \{x, y\}$ for $k = g - 2$ (g is the girth), but a lot of them does not have diameter k. Graphs with these properties are the so-called *goal-minimally k-elongated graphs* (see [13]). After realizing these observations Plesník [17] formulated a conjecture:

Conjecture [Plesník]. If a (δ, g)-cage has diameter $g - 2$, then it is a $(g - 2)$-GMD graph.

5 Conclusion and open problems

The current situation about k-GMD graphs shows that we are still far away from the solution of Kyš's conjecture, because they are known only for 22 distinct values of k. It would be a great success to design a construction which would give k-GMD graphs for infinitely many values of k. A partially success would be to solve some of the above mentioned open problems.

Acknowledgment

The author would like to thank Ján Plesník for valuable discussion on the subject.

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The packing chromatic number of distance graphs

Přemek Holub

with J. Ekstein and B. Lidický

The packing chromatic number $\chi_\rho(G)$ of a graph G is the smallest integer k such that vertices of G can be partitioned into disjoint classes $X_1, ..., X_k$ where vertices in X_i have pairwise distance greater than i. The concept of packing coloring comes from the area of frequency planning in wireless networks. This model emphasizes the fact that some frequencies are used more sparely than the others.

The very first results about packing chromatic number were obtained by Slopper [7]. He studied an *eccentric coloring* but his results were directly translated to the packing chromatic number. The concept of packing chromatic number was introduced by Goddard et al. [5] under the name *broadcast chromatic number*. The term packing chromatic number was later proposed by Brešar et al. [1]. The determination of the packing chromatic number is computationally difficult. It was shown to be $\mathcal{NP}$-complete for general graphs in [5]. Fiala and Golovach [3] showed that the problem remains $\mathcal{NP}$-complete even for trees.

The research of the packing chromatic number was driven by investigating $\chi_\rho(\mathbb{Z}^2)$ where $\mathbb{Z}^2$ is the Cartesian product of two infinite paths - the (2-dimensional) *square lattice*. Goddard et al. [5] showed that $9 \leq \chi_\rho(\mathbb{Z}^2) \leq 23$. Fiala et al. [4] improved the lower bound to 10 and Holub and Soukal [6] improved the upper bound to 17. The lower bound was pushed further to 12 by Ekstein et al. [2].

Let $D = \{d_1, d_2, ..., d_k\}$, where d_i are positive integers and $i = 1, 2, ..., k$. The (infinite) *distance graph* $G(\mathbb{Z}, D)$ with distance set D has the set $\mathbb{Z}$ of integers as a vertex set and in which two distinct vertices $i, j \in \mathbb{Z}$ are adjacent if and only if $|i - j| \in D$.

The study of a packing coloring of distance graphs was initiated by Togni [8]. For large values of t Togni proved the following theorem.

Theorem 1 [8]. *For every $q, t \in \mathbb{N}$:*

$$\chi_\rho(D(1,t)) \leq \begin{cases} 86 & \text{if } t = 2q+1, q \geq 36, \\ 40 & \text{if } t = 2q+1, q \geq 223, \\ 173 & \text{if } t = 2q, q \geq 87, \\ 81 & \text{if } t = 2q, q \geq 224, \\ 29 & \text{if } t = 96q \pm 1, q \geq 1, \\ 59 & \text{if } t = 96q + 1 \pm 1, q \geq 1. \end{cases}$$

We improve some results of Theorem 1 as follows.

Theorem 2. *For any odd integer $t \geq 575$,*

$$\chi_\rho(D(1,t)) \leq 35.$$

For any even integer $t \geq 648$,

$$\chi_\rho(D(1,t)) \leq 56.$$

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LARGE VERTEX-TRANSITIVE GRAPHS OF GIVEN DEGREE AND DIAMETER
Extended Abstract

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We use the term (k,g)-*graph* to denote a (finite, simple) k-regular graph of girth g. A (k,g)-*cage* is a smallest k-regular graph of girth g; its order is denoted by $n(k,g)$. The value of $n(k,g)$ is known for a very limited set of parameters (k,g), and we refer to the problem of finding the (k,g)-cages and determining the value of $n(k,g)$ as the *Cage Problem* [3].

A (Δ, D)-graph is similarly a (finite, simple) Δ-regular graph of diameter D. The *Degree/Diameter Problem* is the problem of determining the order $n_{\Delta,D}$ of the largest (Δ, D)-graphs. As is the case for cages, the order of the largest (Δ, D)-graphs is only determined for very limited classes of parameters [9].

The well-known *Moore bound* serves simultaneously as a lower bound on the order of cages and as an upper bound on the order of (Δ, D)-graphs. In terms of k and g, it can be stated as follows:

$$M(k,g) = \begin{cases} 1 + k + k(k-1) + ... + k(k-1)^{(g-3)/2}, & g \text{ odd} \\ 2(1 + (k-1) + ... + (k-1)^{(g-2)/2}), & g \text{ even} \end{cases} \qquad (1)$$

Graphs whose order is equal to the Moore bound (for the corresponding parameters) are called *Moore graphs*, but are known to exist for only a few pairs of parameters and for most parameter sets are unattainable. Although the Cage and the Degree/Diameter Problems are often thought of as mutually dual problems tied together through their relation to the Moore bound, the study of the relation between the order of the extremal graphs and the Moore bound $M(k,g)$ is disproportionally more developed on the side of cages [7, 2] and the survey paper [9] specifically states that "Finding better (tighter) upper bounds for the maximum possible number of vertices, given the other two parameters, and thus attacking the degree/diameter problem 'from above', remains a largely unexplored area". One of the aims of our presentation is to address this issue in the case of vertex-transitive and Cayley graphs.

A *vertex-transitive graph* is a graph with an automorphism group that acts transitively on its set of vertices. A *Cayley graph* is a vertex-transitive graph that admits the existence of an automorphism group acting regularly (transitively but with trivial vertex stabilizers) on its vertex set. Vertex-transitive and Cayley graphs play an important role in the Cage and Degree/Diameter Problems; with a significant proportion of the known extremal graphs as well as of the current record holders being vertex-transitive or even Cayley. Thus, the study of vertex-transitive (k, g)- and (Δ, D)-graphs bears the dual benefit of potentially producing new extremal graphs or records as well as improving our understanding of the role of symmetry in the constructions of graphs extremal with respect to the two problems.

Once again, a disparity exists between the level of our knowledge of vertex-transitive (k, g)-graphs and of vertex-transitive (Δ, D)-graphs. Both general vertex-transitive and Cayley graphs are known to exist for any pair of parameters (k, g) [10, 8, 6, 5], but no equivalent constructions are known for given Δ and D. Furthermore, improved bounds (as compared to the Moore bound) have been found for the order of vertex-transitive (k, g)-graphs in [6]:

Theorem 0.1 ([6]) *Let G be a vertex-transitive graph of valence k and girth $g = p^r > k$, where p is an odd prime and $r \geq 1$. If G is not a Moore graph (that is, $|V(G)| > M(k, g)$), and g is relatively prime to all the integers in the union*

$$\bigcup_{0 \leq i \leq k} \mathcal{L}(k, g, i),$$

where $\mathcal{L}(k, g, 0) = \{M(k, g) + 1, M(k, g) + 2, \ldots, M(k, g) + k\}$, and $\mathcal{L}(k, g, i) = \{k(k-1)^{(g-1)/2} - ik,\ k(k-1)^{(g-1)/2} - ik + 1, \ldots,\ k(k-1)^{(g-1)/2} - ik + i - 1\}$, $i > 0$, then the order of G is at least $M(k, g) + k + 1$.

In addition, several relations have been determined that tie together the girth of Cayley graphs of nilpotent or solvable groups and their nilpotency or derived length [1, 4]. For example, the girth of Cayley graphs based on nilpotent groups of nilpotency ν have been shown to be bound from above by ν^2.

In our talk we will present a solution to the disparity between our understanding of vertex-transitive (k, g)- and (Δ, D)-graphs at least for the case of the last two above mentioned problems: the order of vertex-transitive (Δ, D)-graphs as compared to the Moore bound and the relation between the nilpotency/solvability and the diameter of the resulting Cayley or vertex-transitive graphs. The results are surprisingly similar to the case of cages, both in flavor and in the actual bounds obtained. This further reinforces the often cited but poorly understood duality of the Cage and Degree/Diameter Problems.

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Mixed Moore Graphs and Directed strongly regular graphs

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In 1988, A. Duval [2] introduced a generalization of strongly regular graphs to directed graphs. A directed strongly regular graph with parameters (n, k, t, λ, μ) is a directed graph on n vertices with adjacency matrix A that satisfies

$$A^2 = tI + \lambda A + \mu(J - I - A) \quad \text{and} \quad JA = AJ = kJ, \tag{1}$$

where J is the all 1 matrix and I is the identity matrix. Thus every vertex is incident to t undirected edges and there are $z = k - t$ edges directed out from a vertex and z edges directed into the vertex. The number of paths from u to v of length 2 is either λ or μ depending on whether there is an edge from u to v or not.

In recent years several papers on directed strongly regular graphs have appeared (e.g. [3], [4], [6], [7], [8], [10]).

A mixed Moore graph is a directed strongly regular graph with $\mu = 1$ and $\lambda = 0$, i.e., for every pair (u, v) of distinct vertices there is a unique path of length at most 2 from u to v. These graphs were investigated by Bosák [1] and by Nguyen, Miller and Gimbert [5]. It is known that for every k there is a unique mixed Moore graph with $t = 1$ and $n = k(k + 1)$ and for $n = 18, t = 3, k = 4$ there is a unique mixed Moore graph, called the Bosák graph. The first open cases have orders $40, 54, 80, 88, 108$.

We will apply theory and ideas from directed strongly regular graphs to mixed Moore graphs. In particular many directed strongly regular graphs have been constructed as (generalizations of) Cayley graphs (see e.g. [3], [6], [10]), but it is known that they can not be Cayley graphs of abelian groups, see [6] or [9].

Many Mixed Moore graphs also appear as Cayley graphs, including the Bosák graph and a new mixed Moore graph with $n = 108, t = 3, k = 10$. This new mixed Moore graph seems to have some relation to the Bosák graph.

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On 3-arc graphs

Martin Knor

Joint work with Guangjun Xu and Sanming Zhou.

Let G be a graph. Its 3-arc graph, $X(G)$, has vertex set identical with the set of arcs of G. Hence, $|V(X(G))| = 2|E(G)|$. Two arcs of G, say $\overrightarrow{uv}$ and $\overrightarrow{xy}$, correspond to adjacent vertices in $X(G)$ if and only if both (v, u, x) and (u, x, y) are paths of length 2 in G. In such a case $\{u, x\}$ is an edge of G and (v, u, x, y) is a 3-arc. The later gave the name to the graph operator.

3-arc graphs were recently used in the classification and characterization of several families of arc-transitive graphs. They are related to line graphs, second iterated line graphs and to path graphs formed by paths of length two.

On IWONT 2007 in Pilsen, Sanming Zhou proposed to study graph-theoretical properties of 3-arc graphs. We study the connectivity, diameter, independence, domination and colourings. For all these invariants, our intention was to bound the parameter of 3-arc graph $X(G)$ using the corresponding parameter of G.

Obviously, if G has a vertex of degree 1, then $X(G)$ contains an isolated vertex. Even the graphs with minimum degree 2 cause troubles, especially when studying the connectivity and the diameter. On the other hand, if the minimum degree of G is at least 3, then the problems are easier.

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Optimal radial Moore graphs of radius two

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Abstract

Given the values of the maximum degree d and the diameter k of a graph, there is a natural upper bound $M_{d,k}$ for its number of vertices known as the *Moore bound*. Graphs that achieve this bound (Moore graphs) are very rare and this special situation has stimulated research on various relaxations of the Moore graph constraints. One way to do it is by allowing the existence of vertices with eccentricity just one more than the value k they should have. In this context, regular graphs of degree d, radius k, diameter $\leq k+1$ and order equal to $M_{d,k}$ are referred to as *radial Moore graphs*. So far, constructions of these extremal graphs are only known for small values of the degree and the radius. Radial Moore graphs have been classified according to their closeness to a theoretical Moore graph using the *status vector* of the graph.

In this work, we define *optimal radial Moore graphs* as the bests graphs ranked according to the status vector classification. We show an heuristic method based on edges swaps operations that changes a radial Moore graph for another one with 'better' status vector. Finally, we apply such heuristic method for the cases of radius $k = 2$ and degrees $5 \leq d \leq 9$ obtaining in each case a radial Moore graph close related to the optimal one.

Keywords: Moore bound, radial Moore graph, ranking measure, status vector.

1 Introduction

The maximum number of vertices in any graph of specified degree and diameter, denoted $M_{d,k}$, is given by the well-known Moore bound, which states that a graph of order n, maximum degree d, and diameter k satisfies

$$n \leq M_{d,k} = 1 + d + d(d-1) + \cdots + d(d-1)^{k-1}. \qquad (1)$$

Graphs achieving this bound are called *Moore graphs*. In the case of diameter $k = 2$, Hoffman and Singleton [6] proved that Moore graphs exist for $d = 2, 3, 7$ (being unique) and possibly $d = 57$, but for no other degrees. They also showed that for diameter $k = 3$ and degree $d > 2$ Moore graphs do not exist. The enumeration of Moore graphs of diameter $k > 3$ was concluded by Damerell [4], who used the theory of distance-regularity to prove their nonexistence unless $d = 2$, which corresponds to the cycle graph of order $2k + 1$ (an independent proof was given by Bannai and Ito [1]).

The fact that there are very few Moore graphs suggested the study of graphs that are in various senses 'close' to being Moore graphs. This 'closeness' has been usually measured as the difference between the (unattainable) Moore bound and the order of the considered graphs. In this sense, the existence of graphs with small 'defect' δ (order $n = M(d,k) - \delta$) has deserved much attention in the literature (see [8]). Another kind of approach considers relaxing some of the constraints implied by the Moore bound. From its definition, all vertices of a Moore graph have the same degree (d) and the same eccentricity (k). We could relax the condition of the degree and admit few vertices with degree $d + \delta$, as Tang, Miller and Lin [9] did for the directed case. Alternatively, we may allow the existence of vertices with eccentricity just on more than the value k they should have. In this context, regular graphs of degree d, radius k, diameter $\leq k+1$ and order equal to $M_{d,k}$ are referred to as *radial Moore graphs*[1].

Figure 1 shows all radial Moore graphs of radius $k = 2$ and degree $d = 3$. The existence of radial Moore graphs has been proved for radius $k \leq 3$ and any degree (see [5], [7]). So far, only a few radial Moore graphs have been found for other values of the degree d and the radius k; more precisely, for (d,k) equal to $(3,4)$, $(4,4)$, $(5,4)$ and $(3,5)$. Besides, the complete enumeration of these extremal graphs is known for the cases $(3,2)$, $(3,3)$ and $(4,2)$. Capdevila et al. (see [3]) provide a measure to rank the population of radial Moore graphs according to their 'proximity' to a theoretical Moore graph, as we explain in the next section.

2 Status vector measure

Let $G = (V,E)$ be a connected graph. Given two vertices u and v of G, the distance between u and v, $d(u,v)$, is the length of a shortest path joining them. The sum of all distances to a vertex v, $s(v) = \sum_{u \in V} d(u,v)$, is referred to as the *status* of v (see [2]). The *status vector* of G, $\mathbf{s}(G)$, is defined as the vector constituted by the status of all its vertices, given in nondecreasing order; that is, $\mathbf{s}(G) = (s_1, s_2, \ldots, s_n)$, where $s_1 \leq s_2 \leq \cdots \leq s_n$. Depending on the graph, the status vector could be too long to be descrived as a vector, then we write it as a sequence, with multiplicities indicated by superscripts. The *total status* of G, $s(G)$, is the sum of all its status: $s(G) = \|\mathbf{s}(G)\|_1 = \sum_{v \in V} s(v) = \sum_{u,v \in V} d(u,v)$. The status vector $\mathbf{s}_{d,k}$ of a Moore graph or degree d and radius k is the vector of dimension $M_{d,k}$ whose components are all equal to $\sum_{i=1}^{k} i \cdot d(d-1)^{i-1}$. This is due to the fact that if we view the Moore graph from any of its vertices, say v, we see exactly d vertices at distance one from v; $d(d-1)$ vertices at distance two from v; and so on up to distance k, where we have $d(d-1)^{k-1}$ vertices. For any connected regular graph G of degree d and order $M_{d,k}$, we have that $s(v) \geq \sum_{i=1}^{k} i \cdot d(d-1)^{i-1}$, $\forall v \in V(G)$, and equality holds for every vertex if and only if G is a Moore graph (see [3]).

Let us denote by $\mathcal{RM}(d,k)$ the set of all nonisomorphic radial Moore graphs of degree d and radius k. Let $G \in \mathcal{RM}(d,k)$, the *norm status* of G is

$$N(G) = \|\mathbf{s}(G) - \mathbf{s}_{d,k}\|_1$$

that is, $N(G)$ measures the difference between the total status of G and the status corresponding to a Moore graph. We can see $N(G)$ as a measure of 'how far' is a radial Moore

[1]These extremal graphs have also been named *radially Moore graphs*.

graph to being a Moore graph in the following sense: If G and G' are both radial Moore graphs of degree d and radius k such that $N(G) < N(G')$, then G has an status vector 'closer' to $\mathbf{s}_{d,k}$ than the corresponding status vector of G'.

Example 1. *In the particular case of degree $d = 3$ and radius $k = 2$, $\mathcal{RM}(3,2)$ has just five graphs, shown in figure 1. Their status vectors are*

$$\mathbf{s}(G_1) : 15,\ 17^9; \quad \mathbf{s}(G_2) : 15,\ 16^4,\ 17^5; \quad \mathbf{s}(G_3) : 15^4,\ 17^6; \quad \mathbf{s}(G_4) : 15^2,\ 16^8.$$

Taking into account that $\mathbf{s}_{3,2} : 15^{10}$, which corresponds to the status vector of the Petersen graph, we have

$$N(G_1) = 18,\ N(G_2) = 14,\ N(G_3) = 12 \text{ and } N(G_4) = 8$$

Hence G_4 has its status vector closer to the minimum one $\mathbf{s}_{3,2}$.

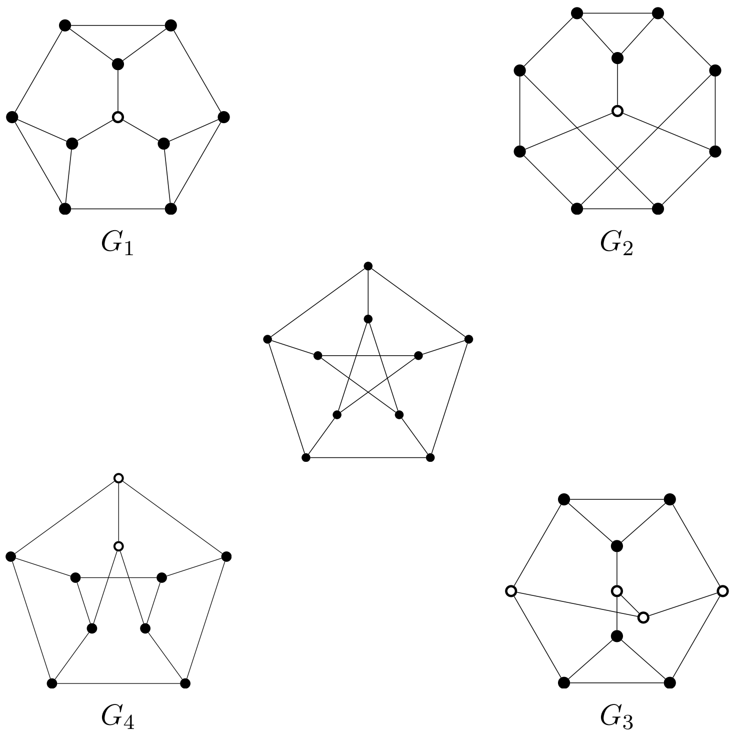

Figure 1: All cubic radial Moore graphs of radius two.

3 Optimal radial Moore graphs

Let $G \in \mathcal{RM}(d,k)$, we say that G is an *optimal radial Moore graph of degree d and radius k* if $N(G) \leq N(H)$, for all $H \in \mathcal{RM}(d,k)$. We denote by $\check{N}(d,k)$ the norm status of an optimal radial Moore graph. Obviously, in the cases (d,k) where Moore graphs exists then $\check{N}(d,k) = 0$. In general, the value of $\check{N}(d,k)$ is known only for a small set of values of (d,k):

there is a unique optimal radial Moore graph for the case $(3,3)$ (and in this case $\check{N}(3,3) = 24$). Besides, $\check{N}(4,2) = 18$ and there are two optimal radial Moore graphs H_4 and $\hat{H}_4$ achieving such value (see figure 2). The graph H_4 belongs to a particular family H_d of radial Moore graphs of degree $d \geq 4$ and radius two where the status vector is known (see [3]):

$$\begin{cases} \mathbf{s}(H_d) = (2d^2 - d)^{2d}, (3d^2 - 6d + 6)^{d^2-2d+1} \\ N(H_d) = (d-1)^2(d-2)(d-3) \end{cases}$$

Captdevila et al. suspect that H_d is farther apart from an optimal radial Moore graph of radius two and degree d, as $d \geq 4$ becomes bigger and they post the problem of finding radial Moore graphs with smaller norm status than H_d. In general, the value of $\check{N}(d,k)$ has been

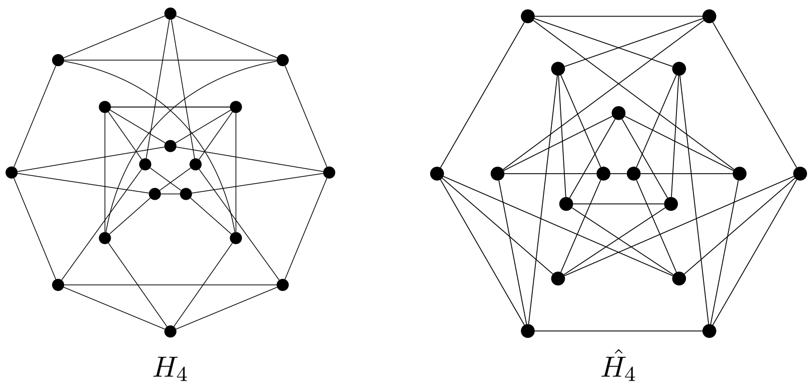

Figure 2: Minimum radial Moore graphs of degree four and radius two. The norm status of both graphs is 18.

determined either in the cases where the Moore graph exists or when the total population of radial Moore graphs is known.

4 An heuristic method to decrease the norm status of a radial Moore graph

In this section we present an heuristic method, based on edges swaps operations, that seeks radial Moore graphs with low norm status. To this end, we start with a radial Moore graph G and let us consider $\mathbf{v}$ as a vector whose elements are the vertices of G ordered according to its status, that is, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ such that $s(v_1) \geq s(v_2) \geq \ldots \geq s(v_n)$. We can think that v_1 is a 'bad' vertex because its high status. We may decrease its status rewiring an edge involving this vertex. Hence, we take w_1 as a neighbor of v_1 with higher status. We do the same with vertex v_2 and we obtain two 'bad' edges (v_1, w_1) and (v_2, w_2) of G. Now, we proceed to rewire as follows: we remove (v_1, w_1) and (v_2, w_2) from G and we add the new edges (v_1, w_2) and (v_2, w_1). So, we obtain a new graph H which differs from G only in two edges. We check if H has radius k and diameter $\leq k+1$ (the two-edges swaps operation preserves the degrees of the vertices) and if it is so, we compute the value of $N(H)$. If $N(H) < N(G)$ we succed, otherwise we can try it with the next pair of neighbors of v_1 and v_2. If every neighbor of v_1 and v_2 gives a graph H such that does not satisfies $N(H) < N(G)$, we follow the next elements in the ordered list $\mathbf{v}$. At the end, either we find a radial Moore graph H

with lower norm status than G or we prove that every graph obtained by G doing two-edges swaps operations has equal or higher norm status than G.

From another point of view, we can see this heuristic method as a movement into the space $\mathcal{RM}(d,k)$. We move from a particular graph to another one if both graphs only differs in a two-edges swaps operation. In fact, these movements are done into a bigger space (the space of regular graphs of degree d on $M_{d,k}$ vertices), but we only admit a movement if in adittion the new graph has radius k, diameter $\leq k+1$ and better norm status.

5 Radial Moore graphs of radius two with low norm status

We take H_d as a basis and we apply iteratively the heuristic method explained above in order to find radial Moore graphs of radius two with low norm status. Sometimes, we can fall in a graph such that any two-edges swaps does not decrease its norm status. To avoid this situation, we admit (with a little probability ϵ) a 'bad' movement into $\mathcal{RM}(d,2)$, that is, ocasionally we may change to a new radial Moore graph with worse norm status than the given graph. Since we already know the optimal graphs for degrees $d=3$ and $d=4$, we start our experiment at $d=5$. Nevertheless, we have applied our algorithm to these small cases and for $d=3$ we have realized that every graph G_1, G_2, G_3 and G_4 moves very fast to the Petersen graph (which is the optimal one). For $d=4$, we pick up at random a radial Moore graph and we fall either to H_4 or H' very fast too. Next, we see what happen for $d \geq 5$:

We start with H_5. In this case, $N(H_5)=96$ and after a few number of algorithm iterations we get a graph H'_5 (depicted in figure 3) such that $N(H'_5)=36$. This graph seems to be unique (we do not find another radial Moore graph with the same norm status value) and it has the lowest value of the norm status that we have reached. Moreover, for any positive even integer $36 \leq k \leq 80$, unless $k=38$, there is a radial Moore graph having k as its norm status (see figure 4). There are other graphs in $\mathcal{RM}(5,2)$ with more central vertices (we have found 3 non-isomorphic graphs containing 11 central vertices), but H'_5 has its status vector closer to the minimum one $\mathbf{s}_{5,2}$ (see table 1). For degree six, we obtain a graph H'_6 with $N(H'_6)=78$ which represents a significant improvement compared to $N(H_6)=300$. We already know

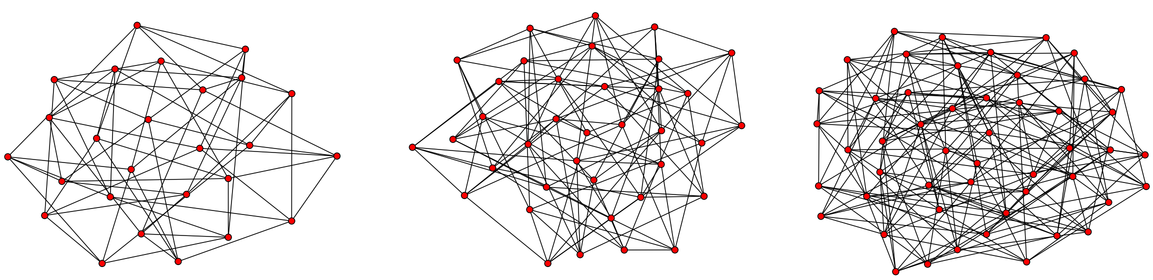

Figure 3: Radial Moore graphs H'_5, H'_6 and H'_7.

the optimal radial Moore graph for the case $(7,2)$, which is the Hoffman-Singleton graph. Starting at H_7 graph ($N(H_7)=720$) we find the Hoffman-Singleton graph in about 1000 iterations (although this number of iterations depends on ϵ). The closest radial Moore graph H'_7 to the Hoffman-Singleton graph found so far has norm status to 40 and it contains 26 central vertices (see figure 4). In addition, the automorphism group of H'_7 has order 40, hence we expect to draw it in the future showing more symmetries. Besides, we have checked that H'_7 is the radial Moore graph with lowest norm status that it can be obtained by the Hoffman-Singleton graph in just two edges-swaps operations. Although we are quite sure

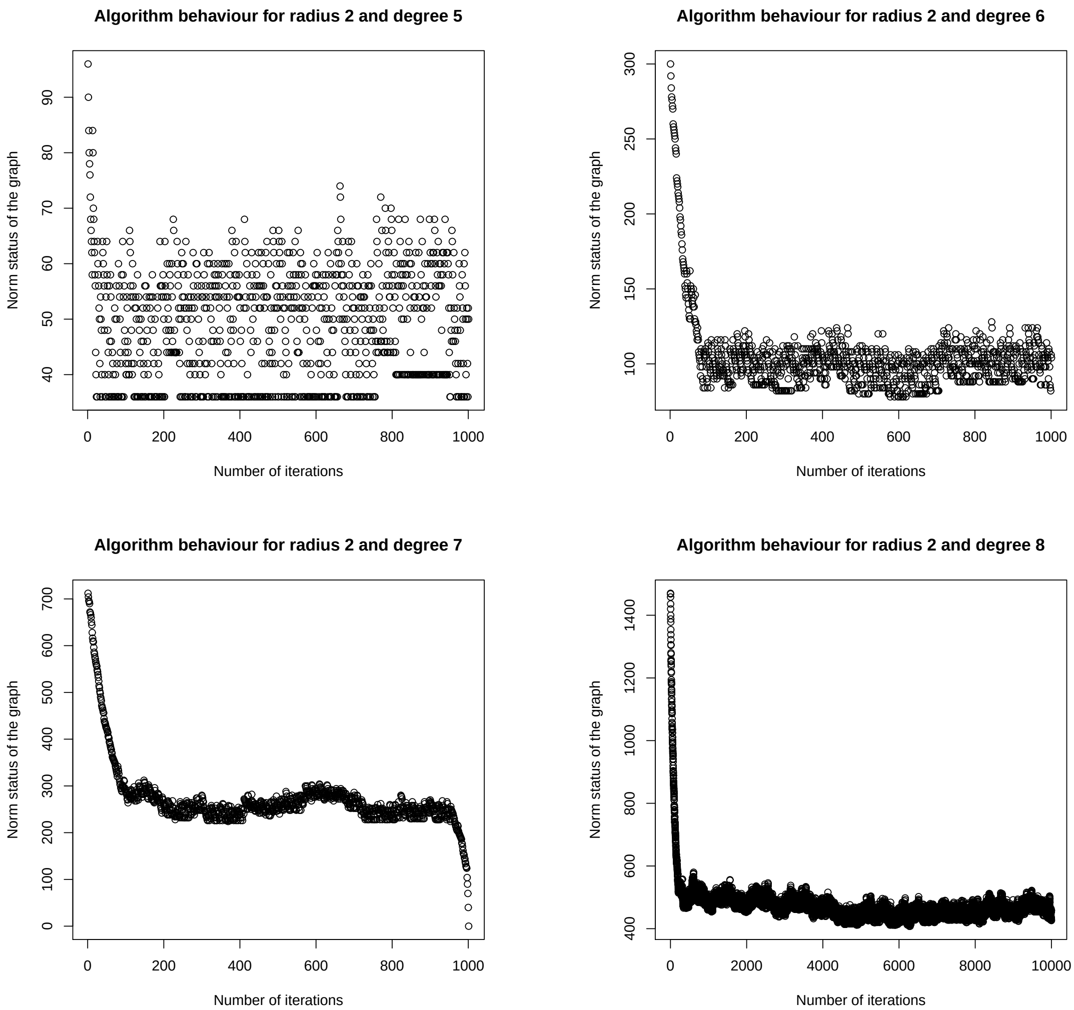

Figure 4: Algorithm behaviour. Each dot in the diagram represents a graph found by our algorithm in $\mathcal{RM}(d, 2)$.

that H'_d are optimal radial Moore graphs for $d \leq 7$, we suspect that H'_d will be farther appart from an optimal one for $d = 8$ and 9. In these cases take an important role the value of ϵ and the number of iterations. For instance, if the algorithm runs 1000 iterations with $\epsilon = 0.01$ the 'best' graph found has norm status equal to 854 for the case of degree 9. Besides, doing 10000 iterations with $\epsilon = 0.0001$ we decrease that value to 784.

	$\mathbf{s}(G)$	$N(G)$
H_5	$45^{10}, 51^{16}$	96
H'_5	$45^6, 46^8, 47^8, 48^4$	36
H_6	$66^{12}, 78^{25}$	300
H'_6	$66^5, 67^9, 68^{13}, 69^5, 71^2, 72^3$	78
H_7	$91^{14}, 111^{36}$	720
H'_7	$91^{26}, 92^{20}, 96^4$	40
H_8	$120^{16}, 150^{49}$	1470
H'_8	$120^1, 123^1, 124^3, 125^{13}, 126^{22}, 127^{13}, 128^7, 129^2, 130^2, 131^1$	408
H_9	$153^{18}, 195^{64}$	2688
H'_9	$153^1, 159^5, 160^8, 161^{11}, 162^{13}, 163^{14}, 164^{14}, 165^{11}, 166^3, 167^2$	784

Table 1: Status vector and norm status for the 'bests' graphs H'_d to be compared with the known values of H_d. For the special case of degree seven, H'_7 is the closest radial Moore graph to the Hoffman-Singleton graph.

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Search for properties of the missing Moore graph

Martin Mačaj and Jozef Širáň

1 Introduction

For any integers $d, k \geq 2$ the order of a graph of maximum degree d and of diameter k is bounded above by the Moore bound $M(d,k) = 1 + d + d(d-1) + \ldots + d(d-1)^{k-1}$; such graphs of order *equal* to $M(d,k)$ are known as the *Moore* (d,k)-graphs. It is well known that, for $d, k \geq 2$, Moore (d,k)-graphs exist only if $k = 2$ and $d = 2$, 3, 7, and possibly 57. In the first three cases the Moore graphs are unique – the 5-cycle, the Petersen graph, and the Hoffman-Singleton graph. For $k = 2$ and 3 this has been known since the pioneering paper by Hoffman and Singleton [5] and for $k \geq 4$ the result was proved independently by Bannai and Ito [2] and Damerell [4].

The aim of this contribution is to investigate possible symmetries (in conjunction with other properties) of the Moore $(57,2)$-graph(s) the existence of which is still in doubt. Throughout, let Γ be a Moore $(57,2)$-graph and let G be its automorphism group.

The study of G was initiated by Aschbacher [1] by proving that G cannot be a rank 3 group. Later in a series of lectures for his graduate students, Graham Higman showed that Γ cannot be vertex-transitive; see Cameron's monograph [3] for an account of the proof. The same argument shows that the order of G is not divisible by 4. This was taken further by Makhnev and Paduchikh [7] by a closer investigation of the structure of G, assuming that G contains an involution. A consequence of their investigation is the bound $|G| \leq 550$ if G has even order.

With the help of a combination of spectral, group-theoretic, combinatorial, and computational methods we prove that $|G|$ can assume only a very restricted set of values. As a corollary we obtain the inequality $|G| \leq 375$ with no restriction on the parity of $|G|$.

2 Ingredients

Throughout, let X be an arbitrary subgroup of G and let A be the adjacency matrix of Γ. It is known that the eigenvalues of A are 57, 7, and -8.

We recall that Higman's approach was backed by the following result.

Theorem 1 *Let V_0, V_1, V_2 be the eigenspaces of A for eigenvalues 57, 7, and -8, respectively. Let χ_0, χ_1 and χ_2 be characters of restrictions of X onto V_0, V_1 and V_2, respectively. For $x \in X$ let $a_i(x) = |\{v \in \Gamma;\ d(v, v^x) = i\}|$, $i = 0, 1, 2$. Finally, let*

$P = \begin{pmatrix} 1 & 1 & 1 \\ 57 & 7 & -8 \\ 3192 & -8 & 7 \end{pmatrix}$ *and* $Q = \frac{1}{3250}\begin{pmatrix} 1 & 1 & 1 \\ 1729 & 637/3 & -13/3 \\ 1520 & -640/3 & 10/3 \end{pmatrix}$. *Then* $Q = P^{-1}$ *and* $(\chi_0(x), \chi_1(x), \chi_2(x))^T = Q(a_0(x), a_1(x), a_2(x))^T$.

We have a number of additional observations regarding the parameters a_i and begin with stating the ones that are of combinatorial nature.

Lemma 1 *For any* $x \in X$ *we have* $a_1(x) \equiv 2a_0(x) \mod 5$, $a_1(x) \equiv a_0(x) + 2 \mod 3$, *and* $a_1(x) \equiv 7a_0(x) + 5 \mod 15$. *Also, if* $x, y \in X$ *are such that* $\langle x \rangle$ *and* $\langle y \rangle$ *are conjugate subgroups of* X, *then* $a_i(x) = a_i(y)$, $i = 0, 1, 2$.

Let $B = (b_{i,j})$ be the adjacency matrix of the equitable partition formed by the orbits of X; we call B the *adjacency matrix* of X. Any parameter related to B (such as trace, eigenvalues, etc.) will be related to X as well (and we will speak of a *trace* of X, *eigenvalues* of X, etc.). If O_i is the i-th orbit of X, we will call $b_{i,i}$ the *trace* of O_i. In general, we define the *trace* of a set of vertices S of Γ to be the average degree of the subgraph of Γ induced by S. Observe that if X_i is the vertex stabilizer of an element $o \in O_i$ under X such that $\mathrm{Fix}(X_i) \cap O_j = \emptyset$ for some $j \neq i$, then $|X_i|$ divides b_{ij}.

A selection of important observations in this connection include:

Lemma 2 *Let* X *have* k *orbits on* Γ. *Then* $\mathrm{Tr}(X) \equiv -8(k-10) \mod 15$. *Further, for any orbit* O *of* X *and any* $v \in V$ *we have* $\mathrm{Tr}(O) = |\{x \in X;\ v \sim v^x\}||O|/|X|$. *Finally,* $|X|\,\mathrm{Tr}(X) = |\{(x, v) \in X \times \Gamma;\ v \sim v^x\}| = \sum_{x \in X} a_1(x)$.

An element $x \in X$ is said to *contribute* to an orbit O of X if $v \sim v^x$ for some $v \in O$.

Lemma 3 *An element* $x \in X$ *contributes to* O *if and only if* x^{-1} *contributes to* O. *Further, if* $|X|$ *is odd then* $\mathrm{Tr}(X)$ *is even, if* x *is central in* X *then* $\mathrm{Tr}(O) \leq 2$, *and for any orbit* O *of* X *we have* $\mathrm{Tr}(O)^2 < |O|$.

We also use a consequence of Mohar's lemma [8].

Lemma 4 *For any* $S \subseteq V(\Gamma)$ *we have* $-8 + |S|/50 \leq \mathrm{Tr}(S) \leq 7 + |S|/65$. *Consequently, for any* $x \in X$ *we have* $a_1(x) \leq 500$.

Besides spectral ingredients we also use structural information about subgraphs of Γ fixed by X to derive information about X, as initiated in [1] and extended in [7]. We omit details in this extended abstract and state just the following:

Lemma 5 *Let* X *be a* p*-group,* $p > 5$. *Then* $\mathrm{Fix}(X) = \emptyset$ *and* $X \cong \mathbb{Z}_{13}$, *or* $\mathrm{Fix}(X)$ *is a single vertex and* $X \cong \mathbb{Z}_{19}$, *or* $\mathrm{Fix}(X)$ *is a pentagon and* $X \cong \mathbb{Z}_{11}$, *or* $\mathrm{Fix}(X)$ *is a star on* $2 + 7l$ *vertices and* $X \cong \mathbb{Z}_7$, *or else* $\mathrm{Fix}(X)$ *is an edge and* $X \cong \mathbb{Z}_7 \times \mathbb{Z}_7$.

A cornerstone for our investigation is a stronger version of Theorem 1 which we present next.

Proposition 1 *Let X be an automorphism group of a Moore $(57,2)$-graph Γ and let χ_0, χ_1, and χ_2 be as defined in Theorem 1. Let $x_1, x_2, \ldots, x_u$ be representatives of equivalence classes of the equivalence $\sim$ defined by $x \sim y$ if and only if $\langle x \rangle$ and $\langle y \rangle$ are conjugate subgroups of X. Let $R_1, R_2, \ldots, R_u$ be irreducible $\mathbb{Q}$-modules of X with characters $r_1, r_2, \ldots, r_u$. Then the system of linear equation with the matrix*

$$\begin{pmatrix} r_1(x_1) & r_2(x_1) & \ldots & r_u(x_1) & | & \chi_i(x_1) \\ r_1(x_2) & r_2(x_2) & \ldots & r_u(x_2) & | & \chi_i(x_2) \\ \vdots & \vdots & \vdots & \vdots & | & \vdots \\ r_1(x_u) & r_2(x_u) & \ldots & r_u(x_u) & | & \chi_i(x_u) \end{pmatrix}$$

has a solution in $\mathbb{N}_0^u$ for any $i \in \{0,1,2\}$.

With the help of all the stated results and some ad-hoc arguments we have determined, for example,

- the feasible groups X acting semiregularly on $V(\Gamma) \setminus \operatorname{Fix}(X)$, including their possible values of traces,
- all possible values of $a_1(x)$ and $\chi_1(x)$ for an element x of odd prime order, and
- all feasible values of $a_1(x)$, $a_1(x^p)$ and $\operatorname{Tr}(x)$ for an element x of order pq for any odd primes $p \leq q$.

3 The strategy

A coarse overview of our strategy of deriving information about groups acting on Γ is as follows:

- Take a group X which one wants to test against possible containment in G.
- Choose the (assumed) values of $a_0(x) = |\operatorname{Fix}(x)|$ that satisfy arithmetic conditions imposed by the information about fixed subgraphs. (An example of such a situation is the fact proved in the 'structural information' part that if $\mathbb{Z}_{25}$ fixes a pentagon, then each element of order 5 must fix a subgraph isomorphic to the Hoffman-Singleton graph.)
- With the help of Lemma 1, determine the possible values of $a_1(x)$.
- Exclude the possibilities that contradict some restriction proved in the 'ingredients' part – this may apply to individual values of $a_1(x)$ as well as to $\operatorname{Tr}(X) = \sum a_1(x)/|X|$.

- If no possibility survives then X with the chosen values of $a_0(x)$ cannot occur. In the remaining cases we usually end up at least with better estimates of the trace than the one from Lemma 2.

The middle step (application of Lemma 1), however, does not make the full advantage of spectral theory. Namely, on the basis of Proposition 1, the values of $a_0(x)$ allow to determine orbits of X together with vertex stabilizers. We may therefore replace this step by the following:

- Determine all the $\mathbb{Q}$-irreducible X-invariant subspaces on the linear space $[\Gamma]$ over $\mathbb{Q}$ whose basis is the vertex set of Γ, and for each assignment of these subspaces to the eigenvalues evaluate the corresponding values of $a_1(x)$.

4 Main results

In this section we list a selection of our results for individual candidates for the subgroup X of G. Some of the facts below are quite hard to prove and in a number of cases we have used the GAP software to determine information about X. The analysis usually yielded a wealth of structural information about orbits and stabilizers which we do not include here.

- If X is a 3-group, then $|X| \leq 27$.
- If X is a 5-group, then $|X| \leq 125$. This is a hard result and involves a large number of intermediate steps based on a fine analysis of possible orbit sizes and their traces. By-products of the analysis furnish observations such as: If $\mathrm{Fix}(X)$ is a Hoffman-Singleton graph, then $|X| \leq 5$.
- If X is a Hall $\{2,3,5\}'$ subgroup of G, then $|X| \in \{1,7,11,13,19,49\}$.
- If X is a $\{3,5\}$-group and if P and Q are its Sylow 3- and 5-subgroups, then Q is normal in X; if P is not normal in X then $|P| = 3$ and $Q \in \{\mathbb{Z}_5^2, \mathbb{Z}_5^3, \mathbb{Z}_5^2 \cdot \mathbb{Z}_5\}$.
- If X is a $\{3,7\}$-group, then $|X|$ divides $3 \cdot 49$.
- If X is a $\{5,13\}$-group but not a 5-group, then $|X| = 13$.

Combined with a number of other results targeting $\{p,q\}$-groups, nilpotent groups, and others we finally obtain:

Theorem 2 *Let G be the automorphism group of a Moore $(57,2)$-graph of odd order. Then $|G| \in \{1,3,5,7,11,13,15,19,21,25,27,35,39,45,55,57,75, 125,135,147,171,275,375\}$.*

In conjunction with refinements of results of [7] this gives:

Theorem 3 *Let G be the automorphism group of a Moore $(57,2)$ graph of even order. Then $|G| \in \{2,6,10,14,18,22,38,50,54,110\}$.*

5 Illustration of methods

We present here three illustrative examples of our methods with sketches of proofs.

Lemma 6 *If* $|X| = 3$*, then* $|\operatorname{Fix}(X)| = 10$.

Proof. An element of order 3 cannot contribute to any orbit. Therefore $\operatorname{Tr}(X) = 0$. However if $|\operatorname{Fix}(X)| = 1$ then X has 1084 orbits and by Lemma 2 we have $\operatorname{Tr}(X) > 0$. □

Lemma 7 *If* X *is a* $\{5, 13\}$*-group but not a* 5*-group, then* $|X| = 13$.

Proof. From the results on the possible orders of 5-groups and 13-groups in G it follows that X is nilpotent, with the 13-group being cyclic of order 13. If $|X| > 13$, then X contains a subgroup of order 65. With the help of Proposition 1 applied to an element x of order 65 one can show that $a_1(x) = 65$, $a_1(x^5) = 65$, and $a_1(x^{13}) = 650$. However, the last value contradicts Lemma 4. □

Proposition 2 *Let* X *be a* 5*-group. If* $\operatorname{Fix}(X)$ *is a Hoffman-Singleton graph, then* $|X| \leq 5$.

Proof. Assume that $|X| = 25$. Since X acts semiregularly on $\Gamma \setminus \operatorname{Fix}(X)$ we have 50 orbits of size 1 and 128 orbits of size 25. In the neighborhood of any fixed point of X there are exactly two orbits of size 25, both with trace equal to 0. As X is abelian of odd order, each of the remaining 28 orbits of size 25 has trace equal to 0 or 2. Therefore the trace of X is at most 56. Moreover, $\operatorname{Tr}(X)$ is even and congruent to $-8 \cdot 168 \equiv 6 \mod 15$. It follows that at least one of these 28 orbits has trace equal to 0.

Let us order the orbits of X in such a way that $O_1, \ldots, O_{50}$ are fixed points of X, O_{50+i} and O_{100+i} lie in the neighborhood of O_i ($i = 1, 2, \ldots, 50$), and $\operatorname{Tr}(O_{178}) = 0$. Let $B = (b_{i,j})$ be the adjacency matrix of X. Because Γ is a Moore graph, $b_{178,50+i} + b_{178,100+i} = 1$ for every $i = 1, 2, \ldots, 50$. Therefore

$$\sum_{i=151}^{177} b_{178,i} = 7 \quad \text{and} \quad \sum_{i=151}^{177} b_{178,i}^2 = 31 \ .$$

This is, however, not possible. □

6 Conclusion

We have obtained a number of new results on the possible order and structure of the automorphism group of the Moore $(57, 2)$-graph(s). If this automorphism group has odd order, our results are completely new, and in the case of even order we have improved the earlier results of Makhnev and Paduchikh.

An interesting phenomenon has occured in the investigation of in the possible actions of the elementary abelian group of order 125 with the smallest orbit of size 25. It turns out that in this case, important parameters of this action such as stabilizers of orbits of size 25 and values of the function a_1 allow for an easy and convenient description in the language of projective geometry. Moreover, this description is, in a way, unique up to the action of the respective projective group. For optimists this may be considered to be an argument pointing at a possible existence of a Moore $(57, 2)$-graph.

Full details including proof can be found in [6].

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Extended abstract
The k-restricted edge-connectivity of a product of graphs

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1 Introduction

Extending a given interconnection system to a larger and fault-tolerant one so that the communication delay among nodes of the new network is small enough is a usual objective in network design. Within the frame of Graph Theory, one interesting model for this kind of extension consists of considering a number of copies of a given graph G, connecting these copies somehow in such a way that the requirements of large connectivity and small diameter are satisfied. Such larger graphs were introduced by Bermond et al. [5]: the *product graph* $G_m * G_p$ of two given graphs G_m, G_p can be viewed as formed by $|V(G_m)|$ disjoint copies of G_p, each edge $xy \in E(G_m)$ indicating that some perfect matching between the copies G_p^x, G_p^y (respectively generated by the vertices x and y of G_m) is placed. Observe that, in fact, the symbol $G_m * G_p$ gathers together a number of non-isomorphic graphs, each of which is determined by a different set of $|E(G_m)|$ perfect matchings between pairs of copies of G_p. Moreover, product graphs $G_m * G_p$ can be regarded as generalizations of two well-known families of graphs. On the one hand, *cartesian product graphs* $G_m \Box G_p$ can be still written as $G_m * G_p$ when each edge of the $|E(G_m)|$ perfect matchings connects two copies of the same vertex of G_p. On the other hand, if $G_m \simeq K_2$ then $G_m * G_p$ results in a *permutation graph* $(G_p)^\pi$ —as introduced by Chartrand and Harary in [7]. Among a large number of references on cartesian product graphs or permutation graphs we can outline some particularly interesting papers, as for example [4, 8, 12, 14, 17, 18, 19, 20], where the study of the connectivity of these graphs has been addressed.

This work approaches the connectedness of product graphs $G_m * G_p$ by means of studying

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the *k-restricted edge-connectivity* of these graphs. Given a connected graph G and an integer k such that $1 \leq k \leq \lfloor |V(G)|/2 \rfloor$, a *k-restricted edge-cut of* G is a set $W \subset E(G)$ such that $G - W$ is not connected and all the components of $G - W$ have at least k vertices. Observe that such a k-restricted edge-cut may not exist; for example, a star on at least four vertices has not 2-restricted edge-cuts. Otherwise, when k-restricted edge-cuts exist in a graph G, then it is said to be $\lambda_{(k)}$-connected. In this case, the minimum cardinality of a k-restricted edge-cut of G is denoted by $\lambda_{(k)}(G)$, and called the *k-restricted edge-connectivity of* G (these concepts were introduced by Fàbrega and Fiol [10, 11], even though in a slightly different way). Notice that $\lambda_{(1)}(G) = \lambda(G)$ corresponds to the (standard) *edge-connectivity of* G, and $\lambda_{(2)}(G) = \lambda'(G)$ is known as the *restricted edge-connectivity of* G, introduced by Esfahanian and Hakimi in [9]. Observe also that $\lambda_{(i)}(G) \leq \lambda_{(j)}(G)$ whenever $i < j$. Apart from the existence of $\lambda_{(k)}(G)$, one important question to be considered concerns its upper bounding. In this regard, a theorem due to Zhang and Yuan ([22]) is specially useful, as it gives a condition for $\lambda_{(k)}(G)$ to exist and to be upper bounded by $\xi_{(k)}(G)$, the so-called *k-edge degree of* G. The k-edge degree of a graph G is defined as the minimum cardinality of $\omega_G(B)$ (set of edges with one endvertex in B and the other one not in B) among all the sets $B \subset V(G)$ on k vertices that induce a connected subgraph of G; $\xi_{(1)}(G) = \delta(G)$ is the *minimum degree of* G, and $\xi_{(2)}(G) = \xi(G)$ is known as the *minimum edge-degree of* G. For other interesting results on the k-restricted edge-connectivity of graphs see for example [1, 2, 3, 4, 6, 13, 15, 16, 21].

In what follows we give some conditions on G_m and G_p that ensure that $G_m * G_p$ is $\lambda_{(k)}$-connected for $k \geq 3$, and present bounds for $\lambda_{(k)}(G_m * G_p)$. Going one step further, we give sufficient conditions to guarantee the optimal result $\lambda_{(k)}(G_m * G_p) = \xi_{(k)}(G_m * G_p)$. As one of the objectives of this work is to generalize or extend somehow some previous results obtained in [1, 15] by the author et al. (for product graphs when $k = 2$, and for permutation graphs with $2 \leq k \leq 5$), we will compare these known results with the main theorem of this work. The following theorem brings together those known results.

Theorem 1 *Let G_m and G_p be two connected graphs. The following statements hold.*

[1] If $G_p \neq K_3$ and $\delta(G_p) \geq \Delta(G_m) + 1 \geq 2$, then the product graph $G = G_m * G_p$ is $\lambda_{(2)}$-connected, and

$$\min\{\lambda(G_m)|V(G_p)|,\ (\delta(G_m)+1)\lambda_{(2)}(G_p),\ \delta(G_m)(\delta(G_p)+1)+\lambda_{(2)}(G_p),\ \xi_{(2)}(G)\} \leq$$
$$\leq \lambda_{(2)}(G) \leq \xi_{(2)}(G).$$

[15] If $2 \leq k \leq 5$, G_p is $\lambda_{(k)}$-connected, and $\delta(G_p) \geq k$, then the permutation graph $G = (G_p)^\pi$ is $\lambda_{(k)}$-connected, and

$$\min\{|V(G_p),\ 2\lambda_{(2)}(G_p),\ \delta(G_p) - k + 3 + \lambda_{(k)}(G_p),\ \xi_{(k)}(G)\} \leq \lambda_{(k)}(G) \leq \xi_{(k)}(G).$$

2 The results

The following theorem constitutes the main result of this work.

Theorem 2 *Let G_m and G_p be two connected graphs. Let $k \geq 3$ be an integer, and assume that G_p is $\lambda_{(k)}$-connected. If $\delta(G_m) \geq k$ and $\delta(G_p) \geq \Delta(G_m) + k - 1$, then the graph $G = G_m * G_p$ is $\lambda_{(k)}$-connected and*

$$\min\{\lambda(G_m)|V(G_p)|,\, (\delta(G_m) - k + 3)\lambda_{(k)}(G_p),\, \delta(G_m)(\delta(G_p) - k + 3) + \lambda_{(k)}(G_p),\, \xi_{(k)}(G)\} \leq \\ \leq \lambda_{(k)}(G) \leq \xi_{(k)}(G).$$

Let us compare Theorem 2 with the known results of Theorem 1. The bounds for $\lambda_{(k)}(G_m * G_p)$ obtained in [1] (first item of Theorem 1) coincide with those in Theorem 2 after taking $k = 2$ —even though Theorem 2 only holds for $k \geq 3$. The only difference in this case is that $G_p \neq K_3$ was required in [1], because otherwise $\lambda_{(2)}(G_p)$ does not exist; to prevent from this problem when $k \geq 3$, the existence of $\lambda_{(k)}(G_p)$ is explicitly imposed in Theorem 2. Hence, we can say that the extension from $k = 2$ to $k \geq 3$ in the bounds for $\lambda_{(k)}(G_m * G_p)$ has been achieved successfully in a very natural way. The comparison of Theorem 2 with the result obtained in [15] for permutation graphs (second item of Theorem 1) is not so straightforward. Recall that a permutation graph $(G_p)^\pi$ can be seen as a product graph $G_m * G_p$ with $G_m \simeq K_2$ (hence $\lambda(G_m) = \delta(G_m) = \Delta(G_m) = 1$). We must first accept that the result in Theorem 1 cannot be deduced as a particular case of Theorem 2 for $k \geq 3$, because $\delta(G_m) \geq k$ (hence $\delta(G_m) \neq 1$ when $k \geq 3$) is a condition of Theorem 2; moreover, Theorem 2 is written for all $k \geq 3$, whereas Theorem 1 holds for $k = 2, 3, 4, 5$. Nevertheless, if we only consider the expressions for the bounds of the k-restricted edge-connectivity in both theorems, it turns out that all except one of the contributions to the lower bound of $\lambda_{(k)}(G_p^\pi)$ in Theorem 1 are particular cases of the contributions in Theorem 2; the term $2\lambda_{(k)}(G_p)$ can be obtained from Theorem 2 only if $k = 2$, yielding this theorem a poorer bound for $k = 3, 4, 5$ than the bound in Theorem 1. Clearly then, an open problem to be approached in the future is to relax the conditions of Theorem 2 with the aim of getting it closer to Theorem 1 in the case of permutation graphs.

The following results states, roughly speaking, that if $\lambda_{(k)}(G_p)$ gets its optimal value $\xi_{(k)}(G_p)$, then this optimality is inherited by $G_m * G_p$ provided that the number of vertices of G_p is large enough.

Corollary 3 *Let $k \geq 3$ be an integer, and G_m and G_p be two connected graphs such that $\delta(G_m) \geq k$, $\delta(G_p) \geq \Delta(G_m) + k - 1$, and $|V(G_p)| \geq k(\Delta(G_p) + \delta(G_p) - k - 1) + 2$. Assume that G_p is $\lambda_{(k)}$-optimal, that is, $\lambda_{(k)}$-connected with $\lambda_{(k)}(G_p) = \xi_{(k)}(G_p)$. Then the graph $G_m * G_p$ is also $\lambda_{(k)}$-optimal, that is, $\lambda_{(k)}(G_m * G_p) = \xi_{(k)}(G_m * G_p)$.*

With the following result we still guarantee $\lambda_{(k)}(G_m * G_p) = \xi_{(k)}(G_m * G_p)$ even though G_p needs not be $\lambda_{(k)}$-optimal. To achieve such a goal, some additional constraint on the maximum degree of G_m is required.

Corollary 4 *Let* $k \geq 3$ *be an integer, and* G_m *and* G_p *be two connected graphs such that* $\delta(G_m) \geq k$, $\Delta(G_m) \leq (\delta(G_m) - 2)(\delta(G_m) - k + 3) - \delta(G_m)$, $\delta(G_p) \geq \Delta(G_m) + k - 1$, *and* $|V(G_p)| \geq k(\Delta(G_p) + \delta(G_p) - k - 1) + 2$. *Assume that* $\lambda_{(k)}(G_p) \geq \xi_{(k)}(G_p) - k(\Delta(G_m) - \delta(G_m) + 2)$. *Then the graph* $G_m * G_p$ *is* $\lambda_{(k)}$*-optimal.*

Note that if $\delta(G_m) = k$, Corollary 4 only makes sense if $k \geq 6$ since otherwise the upper bound on $\Delta(G_m)$ is smaller than $\delta(G_m)$.

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On Maximum Size of Graphs with Girth Greater than 8

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Abstract

The *extremal number*, $ex(n;t) = ex(n;\{C_3, C_4, \ldots, C_t\})$, is the maximum number of edges in a graph of order n that contains no cycle C_k such that $k \leq t$. The set of such graphs is called the *extremal* $\{C_3, C_4, \ldots, C_t\}$*-free graphs* (or just *extremal graphs*), denoted $EX(n;t) = EX(n;\{C_3, C_4, \ldots, C_t\})$, where $t \geq 3$. We use the notation $ex_l(n;t)$ and $ex_u(n;t)$ to indicate lower and upper bounds of $ex(n;t)$ when the exact value is not yet known.

The problem of determining the extremal number for $t > 4$ has recently received much attention. The authors of [5], used hybrid simulated annealing and genetic algorithm to produce constructive lower bounds on the function $ex(n;t)$ for $t \in \{5, 6, 7\}$. Abajo and Diánez [2] proved many of the lower bounds published in [5] to be exact as well as establishing some new upper and lower bounds on the extremal number for $t \in \{5, 6, 7\}$. Further results for $t = 6$ were recently given by Delorme *et al.* [3] who established the extremal numbers, $ex(n;6)$ for $n = 29, 30$ and 31, which are 45, 47 and 49, respectively.

Recent work by Abajo and Diánez [1] established the exact values of the extremal number for $t \geq 4$ and $n \leq \lfloor (16t - 15)/5 \rfloor$. Let $k \geq 0$ be an integer. For each $t \geq 2\, log_2\, (k+2)$, there exists n such that every extremal graph G with $m - n = k$ has minimum degree at most 2, and is obtained by adding vertices of degree 1 and/or subdividing a graph or a multigraph

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H with $\delta(H) \geq 3$ and $|E(H)| - |V(H)| = k$. In [1], Abajo and Diánez determined the extremal number $ex(n;t)$ for $n \leq \lfloor \frac{(16t-15)}{5} \rfloor$ for all $t \geq 4$. To our knowledge, these are currently the only published known values for $t \in \{9, 10\}$. The exact value of $ex(n;8)$, for $n \in \{23, 24, 25\}$, and constructive lower bounds for $n \leq 69$ are given in [4].

In this paper we find new lower bounds on the maximum size of graphs with prescribed order n and girth $g > t$, for $t \in \{8, 9, 10\}$ and $n \leq 200$. We use these new bounds and some new constructions to establish the maximum size of the graphs with order; $n \in \{23, 24, 25, 26\}$ and $t = 8$; $n \in \{26, 27, 28, 29\}$ and $t = 9$; and $n = 30$ and $t = 10$. Furthermore, we describe some constructions that produce infinite families of graphs of maximum size, under girth and order restraints.

The task of obtaining new upper and lower bounds and exact values for the extremal number is repetitive and arduous. Such tasks are best done by computer. To this end we have written a computer program that produces improved lower bounds. The program takes a dense graph with girth $g > t$ as a seed and grows dense graphs with higher order. The graphs that we used as seeds are the cycles, cages, the Petersen graph, the Hoffman-Singleton graph, and the graphs obtained in [1].

The new results are displayed in Tables 1, 2 and 3. The new values are shown in italics, exact values are in bold text. If two values are present these are the improved lower and upper bounds. For $n \geq 70$ we have only listed the lower bound but the upper bound can be easily calculated to be $ex_u(n+1;9) \leq ex_u(n;9) + 3$. This is due to the minimum degree being less than or equal to 3 when n is between 62 and 242.

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n	0	1	2	3	4	5	6	7	8	9
0	**0**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**9**
10	**10**	**11**	**12**	**14**	**15**	**16**	**18**	**19**	**21**	**22**
20	**23**	**25**	**27**	***28***	***29***	***30***	***32***	*33-34*	*35-36*	*36-38*
30	*38-40*	*39-42*	*41-44*	*43-46*	*44-48*	*46-50*	*47-52*	*49-54*	*51-56*	*52-58*
40	*54-60*	*55-62*	*57-64*	*59-66*	*60-68*	*62-70*	*64-73*	*66-76*	*68-79*	*69-82*
50	*71*	*72*	*74*	*76*	*78*	*80*	*82*	*84*	*87*	*88*
60	*90*	*91*	*93*	*95*	*97*	*99*	*100*	*102*	*103*	*105*
70	*107*	*108*	*110*	*112*	*114*	*115*	*117*	*118*	*120*	*122*
80	*124*	*125*	*127*	*129*	*131*	*133*	*134*	*136*	*138*	*140*
90	*141*	*143*	*145*	*147*	*149*	*151*	*153*	*155*	*157*	*159*
100	*161*	*163*	*164*	*166*	*168*	*170*	*172*	*173*	*175*	*177*
110	*179*	*181*	*183*	*185*	*187*	*188*	*190*	*192*	*194*	*196*
120	*198*	*200*	*202*	*204*	*206*	*208*	*210*	*212*	*213*	*215*
130	*217*	*219*	*221*	*223*	*225*	*227*	*229*	*231*	*233*	*235*
140	*237*	*239*	*241*	*243*	*245*	*247*	*249*	*251*	*253*	*255*
150	*257*	*259*	*261*	*263*	*265*	*267*	*269*	*272*	*274*	*275*
160	*277*	*279*	*281*	*283*	*285*	*287*	*289*	*291*	*293*	*295*
170	*297*	*299*	*301*	*303*	*305*	*307*	*309*	*311*	*313*	*315*
180	*317*	*319*	*321*	*323*	*325*	*327*	*329*	*331*	*333*	*335*
190	*337*	*339*	*341*	*343*	*345*	*347*	*349*	*351*	*353*	*355*
200	*357*									

Table 1: New results for $ex(n;8)$, $n \leq 26$; and $ex_l(n;8)$, $n \leq 200$.

n	0	1	2	3	4	5	6	7	8	9
0	**0**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
10	**10**	**11**	**12**	**13**	**15**	**16**	**17**	**18**	**20**	**21**
20	**23**	**24**	**25**	**27**	**28**	**30**	***31***	***32***	***34***	***36***
30	*37-38*	*38-40*	*40-42*	*42-44*	*43-46*	*44-48*	*46-50*	*48-52*	*49-54*	*50-56*
40	*52-58*	*54-60*	*55-62*	*57-64*	*59-66*	*60-68*	*61-70*	*63-72*	*64-74*	*66-76*
50	*68-78*	*70-80*	*71-82*	*72-83*	*74-84*	*76-85*	*78-86*	*79-87*	*81-88*	*83-89*
60	*85-90*	*86-91*	*88-92*	*90-93*	*92-96*	*94-99*	*96-102*	*98-105*	*100-108*	*102-111*
70	105	*106*	*107*	*109*	*110*	*112*	*114*	*115*	*116*	*118*
80	*120*	*121*	*123*	*124*	*126*	*128*	*129*	*131*	*133*	*134*
90	*136*	*138*	*139*	*141*	*143*	*144*	*146*	*148*	*149*	*151*
100	*152*	*154*	*156*	*158*	*159*	*161*	*163*	*165*	*167*	*168*
110	*170*	*172*	*174*	*176*	*178*	*179*	*181*	*183*	*185*	*186*
120	*188*	*189*	*191*	*193*	*195*	*196*	*198*	*200*	*202*	*203*
130	*205*	*207*	*208*	*210*	*212*	*214*	*216*	*218*	*220*	*222*
140	*224*	*226*	*228*	*230*	*232*	*234*	*235*	*237*	*239*	*241*
150	*243*	*246*	*247*	*248*	*249*	*250*	*252*	*254*	*256*	*258*
160	*259*	*261*	*263*	*265*	*267*	*269*	*271*	*273*	*275*	*277*
170	*279*	*281*	*283*	*285*	*287*	*289*	*291*	*293*	*295*	*297*
180	*299*	*301*	*303*	*305*	*307*	*308*	*310*	*312*	*314*	*316*
190	*318*	*320*	*322*	*324*	*326*	*328*	*330*	*332*	*334*	*336*
200	*338*									

Table 2: New results for $ex(n;9)$ and $ex_l(n;9)$, $n \leq 200$.

n	0	1	2	3	4	5	6	7	8	9
0	**0**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
10	**9**	**11**	**12**	**13**	**14**	**15**	**17**	**18**	**19**	**20**
20	**22**	**23**	**24**	**26**	**27**	**28**	**30**	**31**	**33**	**34**
30	***35***	*36*	*38*	*39*	*41*	*42*	*43*	*44*	*46*	*47*
40	*48*	*50*	*51*	*52*	*54*	*55*	*56*	*58*	*59*	*61*
50	*62*	*64*	*65*	*67*	*68*	*70*	*71*	*73*	*74*	*75*
60	*77*	*79*	*80*	*82*	*83*	*85*	*86*	*87*	*89*	*90*
70	*91*	*93*	*94*	*96*	*97*	*99*	*100*	*102*	*103*	*105*
80	*106*	*108*	*109*	*111*	*113*	*115*	*117*	*119*	*121*	*123*
90	*125*	*127*	*129*	*130*	*132*	*134*	*136*	*138*	*140*	*142*
100	*144*	*146*	*147*	*149*	*151*	*153*	*155*	*157*	*159*	*161*
110	*163*	*165*	*168*	*169*	*171*	*172*	*174*	*175*	*177*	*178*
120	180	*181*	183	*184*	*186*	*188*	*189*	*190*	*192*	*193*
130	*195*	*196*	*198*	*199*	*201*	*202*	*204*	*205*	*207*	*208*
140	*210*	*211*	*213*	*214*	*216*	*217*	*219*	*220*	*222*	*223*
150	*225*	*226*	*228*	*229*	*231*	*232*	*234*	*236*	*238*	*239*
160	*240*	*242*	*244*	*245*	*247*	*248*	*250*	*251*	*253*	*255*
170	*256*	*258*	*259*	*261*	*262*	*264*	*266*	*267*	*269*	*270*
180	*272*	*273*	*275*	*276*	*278*	*279*	*281*	*282*	*284*	*285*
190	*287*	*289*	*291*	*292*	*294*	*295*	*297*	*298*	*300*	*302*
200	*303*									

Table 3: New results for $ex(n;10)$, $n \leq 200$.

COVERING GRAPHS WITH MATCHINGS OF FIXED SIZE

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1 Introduction

Let m be a positive integer and let G be a graph. An $[m]$-covering of G is a set $\mathcal{M} = \{M_1, \ldots, M_k\}$ of matchings (1-regular subgraphs) of G, each of size m, such that $\cup_{i=1}^{k} M_i = E(G)$, where $E(G)$ denotes the edge-set of G. We will say a graph $[m]$-coverable if G admits an $[m]$-covering. An $[m]$-covering of smallest order will be called excessive $[m]$-factorization of G and the order of any excessive $[m]$-factorization of G will be denoted by $\chi'_{[m]}(G)$ and called excessive $[m]$-index. We set $\chi'_{[m]}(G) = \infty$ if G is not $[m]$-coverable.
In Section 2 we show that the excessive $[m]$-index is strictly related to outstanding conjectures of Berge, Fulkerson and Seymour and we propose a new conjecture of the same type. In Section 3 we summarize the results obtained for the case in which m is small.

2 Matchings of large size

An outstanding conjecture of Berge and Fulkerson is usually states as follows: for each bridgeless cubic graph G there exist six perfect matchings of G with the property that each edge of G is contained in exactly two of them. It is straightforward that the Berge-Fulkerson conjecture implies the existence of five perfect matchings covering the edge-set of G; it is sufficient to select five of the six perfect matchings. I have recently proved in [7] that an equivalent formulation of the Berge-Fulkerson conjecture is the following:

Conjecture 1 *Let G be a bridgeless cubic graph of order $2n$. Then, $\chi'_{[n]}(G) \leq 5$.*

Having in mind this result, we have considered in [1] the case in which G is cubic of order $2n$ and $m = n - 1$, in other words we consider matchings of size one less than a perfect matching. We propose the following conjecture:

Conjecture 2 *Let G be a bridgeless cubic graph of order $2n$. Then, $\chi'_{[n-1]}(G) = 4$.*

There are some large classes of cubic graphs for which the conjecture is verified to be true: among the others we recall 3-edge-colorable graphs, almost Hamiltonian graphs and graphs having oddness 2 or 4.
Furthermore, we are able to construct an infinite family of 1-connected cubic graphs for which the excessive $[n-1]$-index is large as we want.

In the case of r-regular graphs G of order $2n$ and $r > 3$, the excessive $[n]$-index cannot be bounded by any constant as proved in [8]. Seymour conjectures in [10] that the “right“ condition to put on is about the edge-boundary of odd-sized subsets of the vertex-set. He defines an r-graph as an r-regular graph such that every odd-sized subset of the vertex-set has edge-boundary at least r and for this relevant class of graphs proposes a generalization of the Berge-Fulkerson conjecture. I prove in [9] that also in this case the conjecture can be stated in terms of the excessive $[n]$-index of G as follows:

Conjecture 3 *Let G be an r-graph of order $2n$. Then, $\chi'_{[n]}(G) \leq 2r - 1$.*

I also exhibit a class of r-graphs for which the bound $2r - 1$ is reached.

3 Matching of small size

If m is a small integer the more ambitious task of finding a general formula to compute $\chi'_{[m]}(G)$ can be considered. Obviously $\chi'_{[1]}(G) = |E(G)|$ and it is easy to prove that $\chi'_{[2]}(G) = \max\{\chi'(G), \lceil |E(G)|/2 \rceil\}$ where $\chi'(G)$ denotes the chromatic index of G. The case $m = 3$ is completely solved by Cariolaro and Fu in [2]. We need the following definition to state their result: a set S of edges is a splitting set if no two edges in S belong to the same $[m]$-matching of G. We denote by $s(G)$ the maximum cardinality of a splitting set of G.

Theorem 1 *Let G be a $[3]$-coverable graph. Then*

$$\chi'_{[3]}(G) = \max\{\chi'(G), \lceil |E(G)|/3 \rceil, s(G)\}$$

The next step is $m = 4$. In a joint work with Cariolaro we are able to prove a complete result for trees in this case:

Theorem 2 *Let T be a $[4]$-coverable tree. Then*

$$\chi'_{[4]}(T) = \max\{\chi'(T), \lceil |E(T)|/4 \rceil, s(T)\}$$

In the same work, we give a generalization of the concept of splitting set. We call a set S of edges a t-splitting set if no t edges in S belong to the same $[m]$-matching of G. We denote by $s_t(G)$ the maximum cardinality of a t-splitting set of G and we set $S(G) = \max_t s_t(G)$.
Using this more general concept we wonder if the excessive [4]-index of all [4]-coverable graphs can be computed in the following way:

$$\chi'_{[4]}(G) = \max\{\chi'(4), \lceil |E(G)|/4 \rceil, S(G)\}$$

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Nonexistence of graphs with cyclic defect

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Abstract

The *degree/diameter problem* is to determine the largest (in terms of the number of vertices) graphs of given maximum degree and given diameter. General upper bounds, called Moore bounds, exist for the largest possible order of such graphs, digraphs and mixed graphs of given maximum degree Δ and diameter D.

The *Moore bound* for an undirected graph of degree Δ and diameter D is

$$M_{\Delta,D} = 1 + \Delta + \Delta(\Delta-1) + \cdots + \Delta(\Delta-1)^{D-1}$$

A graph that attains the Moore bound is called a *Moore graph*. Moore graphs exist; they are all the complete graphs and odd length cycles plus two or possibly three special graphs when $D = 2$ and $\Delta = 3, 7$, and possibly 57.

Almost Moore graphs are graphs of degree Δ and diameter D with $M_{\Delta,D} - 1$ vertices; such graphs are also called *graphs of defect 1*. The only almost Moore graphs are the even length cycles.

Almost almost Moore graphs, or *graphs of defect 2*, are graphs of degree Δ and diameter D with $M_{\Delta,D} - 2$ vertices. Such graphs have not been categorised yet. Currently there are only five known graphs of defect 2. One of these is the Mobius ladder which has the distinction of being a "graph with cyclic defect".

In a graph G of defect 2, any vertex v can reach within D steps either two vertices (called *repeats* of v) in two different ways each, or one vertex (called *double repeat* of v) in three different ways; all the other vertices of G are reached from v in at most D steps in exactly one way.

The *repeat (multi)graph* of G, $R(G)$, consists of the vertex set $V(G)$ and there is an edge $\{u, v\}$ in $R(G)$ if and only if v is a repeat of u (and vice versa) in G. Clearly, when defect is 2, $R(G)$ is either one cycle of length $n = |V(G)|$ or a disjoint union of cycles whose sum of lengths is equal to n.

If $R(G)$ is cycle of length n then we say that G has *cyclic defect*. Graphs with cyclic defect were first studied by Fajtlowicz [2] who proved that when $D = 2$ the only graph with cyclic defect is the Mobius ladder on eight vertices (with $\Delta = 3$). Subsequently, for $D \geq 3$, Delorme

and Pineda-Villavicencio [1] proposed several ingenious algebraic techniques for dealing with graphs with cyclic defect and they proved the nonexistence of such graphs for many values of D and Δ. They conjectured that graphs with cyclic defect do not exist for $D \geq 3$.

In this talk we show how structural properties of graphs with cyclic defect can be used to prove that this conjecture holds in general.

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On the M–property for distance–regular graphs

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Abstract

We analyze when the Moore–Penrose inverse of the combinatorial Laplacian of a graph is an M–matrix; that is, it has non–positive off–diagonal elements or, equivalently when the Moore–Penrose inverse of the combinatorial Laplacian of a graph is also the combinatorial Laplacian of another network. When this occurs we say that the graph has the M–property. We prove that only distance–regular graphs with diameter up to three can have the M–property and we give a characterization of the graphs that satisfy the M–property in terms of their intersection array of those distance–regular graphs that satisfy the M–property. In addition, we conjecture that no primitive distance–regular graph with diameter three has the M–property.

Keywords: Distance–regular graphs, Moore–Penrose inverse, strongly regular graph, partial geometry, pseudo geometric graph.

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1 Statement of the Problem

Problems in biological, physical and social sciences can very often be reduced to problems involving matrices which have some special structure. One of the most common situation is when the matrix has non positive o diagonal and non negative diagonal entries; that is $L = kI - A$, $k > 0$ and $A \geq 0$, where the diagonal entries of A are less than or equal to k. If k is at least the spectral radius of A, then L is called an *M–matrix*. In the graph theory framework L is the *combinatorial Laplacian of a k–regular graph*, where A is its adjacency matrix.

Although it is well-known that L has a generalized inverse which is non negative, this is not always true for any generalized inverse. In particular, it may happen that the Moore Penrose inverse, $L^\dagger$, has some negative entries. For instance, this happens for the Moore-Penrose inverse of the combinatorial Laplacian of Petersen's graph.

The combinatorial Laplacian, L, is a symmetric positive semi definite matrix that has 0 as its lowest eigenvalue and whose associated eigenvectors are constant. In consequence, $\mathsf{L}^\dagger$ is an M matrix i $\mathsf{L}^\dagger{}_{ij} \leq 0$ for any $x_i, x_j \in V$ with $i \neq j$.

We will say that a graph Γ has the *M–property* i $\mathsf{L}^\dagger$ is an M matrix.

2 Distance–regular graphs with the M–property

Throughout this section we consider a *distance–regular* Γ with intersection array

$$\iota(\Gamma) = \Big\{b_0, b_1, \dots, b_{D-1}; c_1, \dots, c_D\Big\}.$$

In addition, $a_i = k - c_i - b_i$ is the number of neighbours of x_j in $\Gamma_i(x_k)$, for $d(x_k, x_j) = i$. Usually, the parameters a_1 and c_2 are denoted by λ and μ, respectively. For all the properties related with distance regular graphs we refer the reader to [4].

Some well-known examples of distance regular graphs are the *n–cycle*, C_n, with diameter $D = \left\lfloor \frac{n}{2} \right\rfloor$, intersection array $\iota(C_n) = \Big\{2, 1, \dots, 1; 1, \dots, 1, c_D\Big\}$, where $c_D = 1$ when n is odd and $c_D = 2$ when D is even, see [4].

On the other hand, Γ is *bipartite* i $a_i = 0$, $i = 1, \dots, D$, whereas Γ is *antipodal* i $b_i = c_{D-i}$, $i = 0, \dots, D$, $i \neq \left\lfloor \frac{D}{2} \right\rfloor$ and then $b_{\lfloor \frac{D}{2} \rfloor} = k_D c_{\lceil \frac{D}{2} \rceil}$ and Γ is an antipodal $(k_D + 1)$ cover of its folded graph, see [4, Prop. 4.2.2]. Observe that C_n is antipodal i n is even. Distance regular graphs other than bipartite or antipodal are *primitives*.

The following lemma shows that for distance regular graph $\mathsf{L}^\dagger$ can be expressed in terms of its intersection array, see [1, Prop. 4.1].

Lemma 2.1 *Let Γ be a distance–regular graph. Then, for all $i, j = 1, \dots, n$*

$$\mathsf{L}^\dagger{}_{ij} = \sum_{r=d(x_i,x_j)}^{D-1} \frac{1}{nk_r b_r}\Big(\sum_{l=r+1}^{D} k_l\Big) - \sum_{r=0}^{D-1} \frac{1}{n^2 k_r b_r}\Big(\sum_{l=0}^{r} k_l\Big)\Big(\sum_{l=r+1}^{D} k_l\Big).$$

The next result follows from the previous Lemma, the Moore Penrose inverse of L is a M matrix i $\mathsf{L}^\dagger{}_{ij} \le 0$ when $d(x_i, x_j) = 1$, since in this case the first term in the right hand side of the expression takes the greatest value.

Proposition 2.2 *A distance–regular graph Γ has the M–property iff*

$$\sum_{j=1}^{D-1} \frac{1}{k_j b_j}\Big(\sum_{i=j+1}^{D} k_i\Big)^2 \le \frac{n-1}{k}.$$

Corollary 2.3 *If Γ has the M–property and $D \ge 2$, then*

$$\lambda \le 3k - \frac{k^2}{n-1} - n.$$

and hence $n < 3k$.

Inequality $3k > n$ turns out to be a strong restriction for a distance regular graph to have the M property.

In the following result, we generalize the above observation by showing that only distance regular graphs with small diameter can satisfy the M property.

Proposition 2.4 *If Γ is a distance–regular graph with the M–property, then $D \le 3$.*

Proof. If $D \ge 4$, then $k = k_1 \le k_i$, $i = 2, 3$ which implies that

$$3k < 1 + 3k \le 1 + k + k_2 + k_3 \le n,$$

and hence Γ does not have the M property. □

A *strongly regular graph* with parameters (n, k, λ, μ) is a graph on n vertices which is regular of degree k, any two adjacent vertices have exactly λ common neighbours and two non adjacent vertices have exactly μ common neighbours. A strongly regular graph is a distance regular graph with $D = 2$.

Proposition 2.5 *A strongly regular graph with parameters* (n, k, λ, μ) *has the* M*–property iff*

$$\mu \geq k - \frac{k^2}{n-1}.$$

In particular, every antipodal strongly regular graph has the M*–property.*

We recall that if Γ is a primitive strongly regular graph with parameters (n, k, λ, μ), then its complement graph is also a primitive strongly regular graph with parameters $(n, n-k-1, n-2-2k+\mu, n-2k+\lambda)$, which in particular implies that $\mu \geq 2(k+1) - n$.

Corollary 2.6 *If* Γ *is a primitive strongly regular graph, then either* Γ *or* $\overline{\Gamma}$ *has the* M*–property. Moreover, both of them have the* M*–property iff* Γ *is a conference graph.*

Proof. If we define $\overline{k} = n-k-1$, $\overline{\lambda} = n-2-2k+\mu$ and $\overline{\mu} = n-2k+\lambda$, then

$$\overline{k} - \frac{\overline{k}^2}{n-1} = k - \frac{k^2}{n-1}$$

and hence

$$\overline{\mu} \geq \overline{k} - \frac{\overline{k}^2}{n-1} \iff \lambda \geq 3k - \frac{k^2}{n-1} - n \iff \mu \leq k - \frac{k^2}{n-1},$$

where the equality in the left side holds iff the equality in the right side holds. Moreover, any of the above inequalities is an equality iff $\overline{\mu} = \mu$ and $\overline{\lambda} = \lambda$; that is iff Γ is a conference graph. The remaining claims follow from Proposition 2.5. □

If Γ is a bipartite distance–regular graph with $D = 3$, its intersection array is $\iota(\Gamma) = \{k, k-1, k-\mu; 1, \mu, k\}$, where $1 \leq \mu \leq k-1$. On the other hand, if Γ is an antipodal distance–regular graph with $D = 3$, its intersection array is $\iota(\Gamma) = \{k, t\mu, 1; 1, \mu, k\}$, where $\mu, t \geq 1$ and $t\mu < k$. When $t = 1$, these graphs are known as *Taylor graphs*, $T(k, \mu)$.

Proposition 2.7 *A distance–regular graph with* $D = 3$ *has the* M*–property iff*

$$k^2 b_1 \Big(b_2 c_2 + (b_2 + c_3)^2 \Big) \leq c_2^2 c_3^2 (n-1).$$

In particular, if Γ *is bipartite, it satisfies the* M*–property iff* $\frac{4k}{5} \leq \mu \leq k-1$, *whereas if* Γ *is antipodal, it has the* M*–property iff it is a Taylor graph* $T(k, \mu)$ *such that* $k \geq 5$ *and* $\frac{k+3}{2} \leq \mu < k$.

If Γ is a bipartite distance regular graph with $D = 3$ and $\mu < k - 1$, it is well known, see for instance [4], that Γ_3 is also a bipartite distance regular graph with $D = 3$ and intersection array $\iota(\Gamma_3) = \{k_3, k_3 - 1, k - \mu; 1, \overline{\mu}, k_3\}$, where $k_3 = \frac{(k-1)(k-\mu)}{\mu}$ and $\overline{\mu} = \frac{(k-\mu-1)(k-\mu)}{\mu}$.

Corollary 2.8 *If Γ is the bipartite distance–regular graph with $D = 3$ and $1 \le \mu < k - 1$, then either Γ or Γ_3 has the M–property, except when*

$$k - 1 < 5\mu < 4k,$$

in which case none of them has the M–property.

If Γ is the Taylor graph $T(\mu, k)$, it is well-known that the graph Γ_2 is also the Taylor graph $T(k - 1 - \mu, k)$.

Corollary 2.9 *If Γ is the Taylor graph $T(k, \mu)$ with $1 \le \mu \le k - 2$, then either Γ or Γ_2 has the M–property, except when $\mu \in \{m - 2, m - 1, m, m + 1\}$ when $k = 2m$ and $\mu \in \{m - 1, m, m + 1\}$ when $k = 2m + 1$, in which case none of them has the M–property.*

To our knowledge there is no primitive distance regular graphs with $D = 3$ satisfying the M-property.

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The Maximum Degree&Diameter-Bounded Subgraph in the Mesh

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Abstract

The problem of finding the largest connected subgraph of a given undirected host graph, subject to constraints on the maximum degree Δ and the diameter D is a generalization of the Degree/Diameter Problem.

Let $G = (V, E)$ be an undirected graph without loops or multiple edges (called the *host graph*), with n vertices (its *order*), and m edges (its *size*). Our problem is stated as follows:

Problem 1 (Maximum Degree/Diameter Bounded Subgraph, MaxDDBS)**.** *Given a connected undirected host graph G, an upper bound Δ for the maximum degree, and an upper bound D for the diameter, find the largest connected subgraph S with maximum degree $\leq \Delta$ and diameter $\leq D$.*

MaxDDBS is a natural generalization of the well-known Degree/Diameter Problem (DDP), which asks for the largest graph with given degree and diameter [3]. DDP can be seen as MaxDDBS when G is the complete graph K_n for sufficiently large n. Problem 1 was recently introduced in [1], where the various practical applications are discussed, and a heuristic approximation algorithm to solve MaxDDBS is given, since it is computationally hard.

MaxDDBS is closely related to the Degree/Diameter Problem (DDP), stated by Elspas in 1964, which consists of finding the largest graph with a given maximum degree Δ and a given diameter D. Since the order of such a graph cannot exceed the quantity $M_{\Delta,D} = 1 + \Delta + \Delta(\Delta-1) + \cdots + \Delta(\Delta-1)^{D-1}$, called the *Moore bound*, if we take G as the complete graph on $M_{\Delta,D}$ vertices (denoted by $K_{M_{\Delta,D}}$) in Problem 1, we get the Degree/Diameter Problem.

A graph whose order is equal to the Moore bound is called a *Moore graph*. Moore graphs are very rare; they exist only for certain special values of diameter: only when $\Delta = 2$ or $D = 1$ or 2. To be more precise, when $\Delta = 2$, Moore graphs are the odd cycles C_{2D+1} of diameter D; for diameter $D = 1$, Moore graphs are the complete graphs of order $\Delta + 1$, while for diameter $D = 2$, Moore graphs exist for $\Delta = 2, 3, 7$ and possibly 57, but not for other degrees [3]. We denote by $N_{\Delta,D}$ the order of the largest graph that can be constructed with maximum degree Δ and diameter D; the current lower bounds for $N_{\Delta,D}$ are shown in [2].

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A case of special interest is when the host graph G is a common parallel architecture, such as the mesh, the hypercube, the butterfly, or the cube-connected cycles. If there are any constraints on communication time between two arbitrary processors, then MaxDDBS corresponds to the largest subnetwork that can be allocated to perform the computation. The case of the mesh and the hypercube as host graphs were already treated in [1], where some bounds were found for the order of MaxDDBS in a k-dimensional mesh.

In this paper we discuss the case of the mesh as a host graph. We refine the bounds given in [1] for the order of the largest subgraph in arbitrary $k \geq 1$, and we focus on the cases $k = 3, \Delta = 4$ and $k = 2, \Delta = 3$. For those particular cases we give constructions that result in larger lower bounds.

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Computing paths in large Cayley graphs and cryptanalytic applications

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Abstract

We describe *Cayley hash functions*, a cryptographic construction based on Cayley graphs. The security of this construction can be related to a famous conjecture of Babai on the diameter of Cayley graphs.

1 Cryptographic hash functions

Cryptography allows secure and secret communications in untrusted environments. *Hash functions* are a very important cryptographic primitive, useful to construct digital signatures and message authentication codes. Informally, a hash function is a mathematic function that maps any arbitrary-long binary *message* m into a *hash value* h of fixed length:

$$H : \{0,1\}^* \to \{0,1\}^L.$$

The three main security requirements for a hash function are collision resistance, second preimage resistance and preimage resistance. They can be informally formulated as follows:

- *Collision resistance:* it is "hard" to find a couple of messages (m, m') such that $H(m) = H(m')$
- *Second preimage resistance:* given a message m, it is "hard" to find another message m' such that $H(m) = H(m')$
- *Preimage resistance:* given a hash value h, it is "hard" to find a message m such that $H(m) = h$.

In all these informal definitions, "hard" can be given a precise *computational* sense. We refer to textbooks [6] for a more complete introduction to cryptography and hash functions.

2 Cayley graphs, expander graphs

Let G be a finite group and let $S := \{s_0, \ldots, s_{k-1}\} \subset G$ be a set of elements of this group. The *Cayley graph* $\mathcal{G}_{G,S}$ is defined as follows: it has one vertex for each element of G and there is one edge between two elements v_1 and v_2 if and only if $v_2 = v_1 s_i$ for some $s_i \in S$.

Intuitively, an *expander graph* is a sparse graph with strong connectivity properties. The formal definition requires a family of graphs $\{\mathcal{G}_i, i \in \mathbb{N}\}$. It says that there exists a constant c such that for any i and any subset V of $\mathcal{G}_i$ containing less than half of the vertices, we have

$$\frac{|\delta V|}{|V|} \geq c$$

where δV is the set of neighbors of V. One important property of expander graphs is that random walks quickly converge to the uniform distribution.

Expander graphs have a lot of applications in computer science and mathematics. There exists both theoretical and experimental evidence that Cayley graphs are good expander graphs. We refer to the survey [5] for more on the topic.

3 Cayley hash functions

The hash functions commonly used in practice (like the SHA family [1]) have an *ad hoc* design aimed to make the the hash function look somehow like a "random function". In contrast, *Cayley hash functions* are an attempt to base the security of hash functions on (hopefully) hard mathematical problems.

Let G be a finite group and let $S := \{s_0, \ldots, s_{k-1}\} \subset G$ be a set of elements of this group. Suppose that the message m is decomposed into k-digits (into bits if $k = 2$):

$$m = m_1 m_2 \ldots m_N, \qquad m_i \in \{0, \ldots k-1\}.$$

We can define a hash function H as

$$H(m_1 m_2 \ldots m_N) := s_{m_1} \cdot s_{m_2} \cdot \ldots \cdot s_{m_N}.$$

From an efficiency point of view, the computation of this hash function can be parallelized easily. On the other hand, its main security properties can be related to group-theoretical and graph-theoretical properties and problems. For example,

- Preimage resistance corresponds to the *factorization problem in the group*: given an element $g \in G$, find a factorization $g = s_1 \cdot s_2 \cdot \ldots \cdot s_N$ where $s_N \in S$.
- The expander properties ensure that the output of the hash function is "well-distributed" (close to uniformly distributed).

The security of the construction seems to depend on the parameters G and S. For example, collision resistance forbids the use of Abelian groups. Some examples of parameters proposed for this construction can be found in [9, 8, 4].

4 Cryptanalysis - the path-finding problem

Interestingly, the *factorization problem in finite groups* is closely related to a well-known conjecture of Babai on the diameter of Cayley graphs [2]. In fact, the problem can be seen as a *constructive* version of the conjecture. Substantial progress has recently been made on the conjecture but the partial proofs we have are mostly non constructive. In graph-theoretical terms, the factorization problem amounts to finding a *routing algorithm* between any pair of vertices in the Cayley graph. Some examples of parameters for which the problem can be solved "efficiently" are given in [3] and [7]. For essentially all remaining parameters, the problem is still widely open.

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On the achromatic and pseudoachromatic index of the complete graph

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Let $G = (V, E)$ a simple graph. A colouring of its vertices $\varsigma : V \to [k]$ is called *complete* if each pair of different colours $i, j \in [k] := \{1, ..., k\}$ appears in a edge; that is, if there exist $e = uv \in E$ such that

$$\varsigma(e) = \{\varsigma(u), \varsigma(v)\} = \{i, j\}.$$

The *pseudoachromatic number* $\psi(G)$ is the maximum k for which there exist a complete colouring of G (cf. [8]). If the colouring is required also to be proper (i. e., that each chromatic class is independent), then such a maximum is know as the *achromatic number* (cf. [9]) and it will be denoted here by $\alpha(G)$. Clearly,

$$\chi(G) \le \alpha(G) \le \psi(G),$$

where $\chi(G)$ denotes, as usual, the chromatic number of G. Interesting results on these invariants can be found in [4, 5, 7, 10].

We are mainly interested in the pseudoachromatic number $\psi(n) := \psi(L(K_n))$ of the complete graph's line graph -also know as the *pseudoachromatic index* of the complete graph- and its relation with the *achromatic index* $\alpha(n) := \alpha(L(K_n))$.

In this talk, we expose the principal motivation of this research, a deep result due to Bouchet (cf. [6]): Let q be and odd natural number, and let $m = p^2 + p + 1$. A projective plane Π_p of order p exists if and only if $\alpha(m) = pm$.

Also, we expose our work made in this direction:

In a recently paper, my coauthors proved that (cf. [2]) $\psi(n) = q(n+1)$ when $n = q^2 + 2q + 2$ and $q = 2^\gamma$ for $\gamma \in \mathbb{N}$ using also the properties of the projective planes.

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Now, we have shown that (cf. [1]) $\psi(n-a)=\alpha(n-a)=q(n-2a)$ when $n=(q+1)^2$, $q=2^\gamma$ for $\gamma\geq 2$ and $a\in\{0,1,2\}$ using also projective planes.

The lower bound is obtained finding colourings that attains it; these colourings are related with the structure of the projective planes, we recall the basic combinatorial properties of the projective planes:

Given a prime power q, let denote as Π_q the projective plane of order q. Such plane has $n=q^2+q+1$ points and n lines; each line contains $q+1$ points and each point belongs to $q+1$ lines. Moreover, two every pair of points belongs to exactly one line, and every pair of lines intersect in exactly one point.

By other side, for the upper bounds we use two simple functions as following (cf. [1, 10, 11]):

$$\psi(m)\leq \text{máx}\left\{\text{mín}\{f_n(x):=\left\lfloor\frac{n(n-1)}{2(x+1)}\right\rfloor, g_n(x):=2x(n-x-1)+1 : x\in\mathbb{N}\}\right\}.$$

And, also, a detailed anaysis due to Jamison [11] of the two functions of above,

$$\psi(n)\leq\begin{cases} g_n(x) & \text{si } n\in\{4x^2-x,\dots,4x^2+3x-1\}\\ f_n(x) & \text{si } n\in\{4x^2+3x,\dots,4(x+1)^2-(x+1)-1\}\end{cases}$$

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Cayley graphs in the degree-diameter problem

Extended Abstract

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A graph of maximum degree $d \geq 3$ and diameter $k \geq 2$ can have at most $M(d,k) = 1 + d + d(d-1) + ... + d(d-1)^{k-1}$ vertices. The *Moore bound* $M(d,k)$ is, however, known to be met by just two graphs for d and k in the given range, namely for $d = 3$ and $k = 2$ by the Petersen graph, and for $d = 7$ and $k = 2$ by the Hoffman-Singleton graph. For all other values of $d \geq 3$ and $k \geq 2$, except, possibly, $d = 57$ and $k = 2$, we know that the largest possible number $n(d,k)$ of vertices of a graph of maximum degree d and diameter k satisfies $n(d,k) < M(d,k)$. These findings, dating back to the sixties and early seventies of the previous century (cf. [5, 1, 2]), have generated numerous interesting problems, all stemming from the main question about determining, or giving at least reasonable estimates on, the number $n(d,k)$. Literature on this topic counts hundreds of papers and we therefore refer to the survey article [9] for more information about the history and the development in the degree-diameter problem.

Since the order of the Moore bound is enormous even for modest values of d and k, it is not a surprise that attempts to construct large graphs of a given diameter and a given maximum degree have been based on algebraic or geometric structures. This applies also to computer hunting for such graphs, where, in addition, the diameter checking is greatly facilitated if the generated graphs are vertex-transitive, or at least have very few orbits of the automorphism group. Perhaps the simplest and most natural way to meet these requirements is to consider building large *Cayley graphs* of a given degree and diameter. This is the case for both computational results (cf. [6] and references therein; see also [13]) as well as more general constructions.

In our presentation we will give a survey of the existing constructions of large Cayley graphs of given degree and diameter. We will not be just revisiting the existing results regarding the role of Cayley graphs in the degree-diameter problem. Rather more importantly, in the talk we will also give an overview of he methods that have been used in what is now known as he *Cayley version* of the degree-diameter problem, which is finding the largest order of a Cayley graph of a given degree and diameter.

Although orders of the largest Cayley graph of given degree and diameter on an Abelian group cannot come anywhere near the Moore bound, these are interesting from the theoretical point of view and we will present details of the current best constructions [3, 4, 7]. In general, no reason is known why the Moore bound could not be at least asymptotically approached by Cayley graphs of non-Abelian groups; a recent example is [11] where, for diameter 2 and an infinite set of degrees d, it is shown that the ratio of the order of the largest Cayley graphs with these parameters and $M(d, 2) = d^2 + 1$ tends to 1. Other available methods of construction of large Cayley graphs of given degree and diameter [7, 8, 10, 11, 12] will be discussed in some details as well.

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Automorphisms on Almost Moore Digraphs

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1 Introduction

Let G be a digraph with degree d, diameter k and order $\sum_{i=1}^{k} d^i$, such a graph is called an almost Moore digraph. For all vertices $v \in V(G)$ there exists exactly one vertex $r(v)$, the repeat of v, for which there is exactly two paths of length at most k from v to $r(v)$ and from v to every other vertex there is exactly one such path. We know that $r : V(G) \mapsto V(G)$ is an automorphism, see [1]. If $v = r(v)$, then v is called a selfrepeat and G contains a selfrepeat if and only if it contains a k-cycle, see [2].
The automorphism r has been studied to some extent, along with the automorphisms r^p defined by $r^0(v) := v$ and $r^p(v) := r(r^{p-1}(v))$ for all natural numbers p and $v \in V(G)$. The order $\omega(v)$ of a vertex v is the smallest integer $p > 0$ such that $r^p(v) = v$.

2 Automorphisms on G

We consider more general automorphisms on almost Moore digraphs than those mentioned above, which we expect will help us characterize further properties of almost Moore digraphs.

For instance, by studying an automorphism φ which fixes at least three vertices, we obtain the following theorem

Theorem 1. *Let G be an almost Moore digraph with $d \geq 4$, $k \geq 3$ and no selfrepeats. Let φ be an automorphism which fixes at least three vertices of G and let $u = \varphi(u)$. Then the vertices which are fixed by φ forms a cycle of length $k + 2$ or $N^+(u)$ contains at least two vertices fixed by φ.*

A theorem by Baskoro and Amrullah [3, Theorem 1] states that every vertex of the smallest order p in an almost Moore digraph without selfrepeats and no vertices of order 2, has at least two out-neighbours of order p. Theorem 1 above is a more general result than that, as φ which fixes at least three vertices is a more general automophism than r^p with $p > 2$.

The properties characterized by the automorphisms we study might also help us prove the existence or non-existence of certain almost Moore digraphs.

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Complete characterization of graphs with order n and metric dimension $n-2$ *

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Abstract

For an ordered set $W = \{w_1, w_2, \cdots, w_k\}$ of vertices and a vertex v in a connected graph G, the representation of v with respect to W is the ordered k-tuple $r(v|W) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$ where $d(x, y)$ represents the distance between the vertices x and y. A set R is called a *resolving set of* G if for every vertex v of G, its representation with respect to R is unique. A resolving set of G is called *basis* of G if it has minimum cardinality among all resolving sets of G. The *metric dimension of* G*, dim(G)*, is the cardinality of a basis of G.

To date, complete characterizations of graphs with order n are known only for dimension 1, $n-1$, and n. In this presentation, we completely characterize graphs of order n and metric dimension $n-2$.

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On directed $(k, 4)$-cage graph.

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Abstract

Let $G = (V, E)$ be a k-regular directed graph. A girth is the minimum length of directed cycle in G. The problem is to find the minimal number of vertices of the directed graph G that has degree k and girth g. The directed graph that fulfill this condition is called directed (k, g)-cage graph. The upper bound of the number of the vertices has been known by the k-regular directed graph of girth g, $\overrightarrow{C}^k_{(g-1)k+1}$. In this paper we give survey on the knowm results in this problem and prove that the upper bound has been reach for some cases on girth 4.

Keywords: regular directed graph, girth

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N-separators in planar networks as a characterization tool

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1 Abstract

We study the N-separators in weighted (vertices, edges and faces) planar graph (an N-*separator* of a connected graph G is a subgraph G whose deletion decomposes G into N connected components). A number of papers were inspired by the original paper by Lipton and Tarjan [6] on 2-separators in weighted (only vertices and edges) planar graphs. Their separator construction is very important issue in many graph applications such as VLSI modelling [7], communication networks [8], parallel computing [9]. The most complete and recent survey on graph separators can be found in [10].

One of the possible applications of the separator method is the degree-diameter problem in planar graphs [11]. In this case, the separator method is used for planar graph characterization [5]. The largest graphs in the degree-diameter problem are very dense. Therefore, the simplest Lipton and Tarjan separator (Lemma 2 of [6]) which is a cycle obtained by addition of an edge to some edges of a spanning tree becomes an efficient tool [2] in the degree-diameter problem. Each of two separated subsets is either the interior or the exterior of the cycle. Further progress can be achieved in the degree-diameter problem increasing the number of separated subsets. In general, an N-separator is needed consisting of several cycles.

We optimize the separator construction in plane graphs with weighted vertices, faces and edges. Such generalization is important for practical applications. We consider the problem of existence of an N-separator in a planar graph and give optimal bounds to the minimum weight component.

We consider the degree-diameter problem restricted to planar graphs. We look for the largest number of vertices $p(\Delta, D)$ in a planar graph with maximum degree Δ and even diameter $D = 2d$. Hell and Seyffart [1] have computed $p(\Delta, 2) = [3\Delta/2] + 1$ and proved that this value is exact for $\Delta \geq 8$. Fellows, Hell, and Seyffarth have also found [2] rather rough upper bounds $p(\Delta, 2d) = (12d + 3)(2\Delta^d + 1)$ for $d > 1, \Delta \geq 4$. To this end they have applied the Lipton and Tarjan separator theorem [6]. Later [3], they have constructed plane graphs proving the lower bound

$$p(\Delta, 2d) = \frac{(3\Delta-4)\Delta(\Delta-1)^{d-1}-4}{2(\Delta-2)} .$$

At the same time they emphasized "that the lower bounds are likely to be closer to the actual values of $p(\Delta, D)$ and that good upper bounds likely to be difficult to establish." They asked as well the question: "Let D be fixed. Is it the case that for all sufficiently large Δ there are networks with maximum degree Δ , diameter at most D, and $p(\Delta, D)$ nodes which are all of the same type?" We improve the constructions of Hell and Seaffart increasing in the case $\Delta \geq 5$ the lower bound:

$$p\left(\Delta, 2d\right) = \left\lfloor \frac{3\Delta}{2} \frac{\left(\Delta - 1\right)^{d} - 1}{\Delta - 2} \right\rfloor + 1 \tag{1}$$

We show that this bound is exact for large Δ: The proof is based on a 5-separator construction in a plane graph. The existence of an N-separator in a plane graph with bounded number of vertices in each face was proved recently [4].

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(2)-pancyclic graphs

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In the widely known book on graph theory (more precisely, its 1976 edition) by J. A. Bondy and U. S. R. Murty [1] we nd a series of 50 open problems, among which problem number 10 will be the initial point of our investigations: Determine all graphs having exactly one cycle of each length p, $3 \leq p \leq n$, where n is the order of the graph (such graphs are called *uniquely pancyclic*). [1] attributes this problem to R. C. Entringer, who formulated it in 1973.

Constructing the four smallest uniquely pancyclic graphs (they are of order 3, 5, 8, and 8) is an easy task. In 1986, Y. Shi [4] constructed three further such graphs (each of order 14), conjecturing that there are no other uniquely pancyclic graphs than these seven. This problem is still widely open, and only recently K. Markstrom [3] con rmed Shi's conjecture for $n \leq 59$.

In this talk we will discuss the class of (2)-*pancyclic* graphs, which are graphs of order n having exactly two cycles of length p for all p ful lling $3 \leq p \leq n$. Very little is known concerning these graphs. We provide examples of such graphs (most of which were constructed by G. Exoo [2]), establish their existence or non-existence for all orders up to 11, and provide all non-isomorphic (2)-pancyclic graphs of smallest order.

We also give bounds on the vertex-degrees in such graphs, present a result exhibiting how many cycles a given edge traverses, and prove a lower bound for the order of non-Eulerian (2)-pancyclic graphs. Furthermore, we introduce (mimicking previous approaches, see e.g. [5]) r-(2)-*pancyclic* graphs, which are graphs of order n featuring exactly two cycles of each length p, $r \leq p \leq n$, and construct an in nite family of such graphs with non-trivial r. Finally, we present a theorem yielding as corollary the existence of (2)-pancyclic digraphs of every order n, $n \geq 3$.

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Extended abstract
A family of large vertex-transitive graphs of diameter 2

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1 Introduction

The interest in large graphs and digraphs of given degree and diameter comes from possible applications in the design of interconnection networks. A closely related problem is the "degree-diameter problem", which is determining the largest graphs and digraphs of given degree and diameter. History and development of this area of research has been summed

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up in the survey [10]. In what follows we focus on a special case - the large graphs of given degree d and diameter 2.

Let $n(d,2)$ be the largest order of a graph of maximum degree d and diameter 2. The upper bound $n(d,2) \leq 1+d+d(d-1) = d^2+1$ is known as the Moore bound for diameter two. The equality $n(d,2) = d^2+1$ holds if and only if $d = 2,3,7$ and possibly 57, see Hoffman and Singleton [8]. The corresponding extremal graphs are called Moore graphs. For all the remaining degrees d we have $n(d,2) \leq d^2-1$ by [6]. The best lower bound for degrees of the form $q+1$ where q is a prime power comes from the Brown's graphs [4] and their extended version [5, 6]. The Brown's graphs, known as polarity graphs, were also studied in [1, 3].

From the computational point of view the most interesting are vertex transitive and Cayley graphs, because of efficient computer generation and fast diameter checking. In this contribution we focus only on vertex-transitive version of this problem and on diameter 2. Let $vt(d,2)$ be the largest order of vertex-transitive graphs of degree d and diameter 2. There is no better upper bound as in the case of $n(d,2)$. The equality $vt(d,2) = d^2-1$ holds for $d = \{2,3,7\}$, and we have $vt(d,2) \leq d^2-1$ for all other degrees, including 57.

The current best lower bound on $vt(d,2)$ was obtained with the construction by McKay, Miller and Širáň [9] and has the form $vt(d,2) \geq 8/9(d+1/2)^2$ for all d such that $d = (3q-1)/2$, where q is a prime power congruent with 1 mod 4. The corresponding graphs have been known as the McKay-Miller-Širáň graphs; they are vertex-transitive but not Cayley. Širáň, Šiagiová and Ždímalová [12] extended this bound for other degrees.

A simplified version of the McKay-Miller-Širáň graphs described by lifts was given by Šiagiová [11]. Hafner [7] and Arraucho, Noy, Serra [2] have given an alternative geometric description of these graphs as modified incidence graph of an affine plane.

2 The result

Let $q = p^n$ be a power of a prime p, let $F = GF(q)$, and let B_q be the bipartite graph of order $2q^2$ with vertex set $V_0 \cup V_1$ where $V_0 = \{(a,x)_0;\ a,x \in F\}$ and $V_1 = \{(b,y)_1;\ b,y \in F\}$ and where the adjacency, that is, the edges set $E(B_q)$ of B_q, is defined by

$$(a,x)_0 \sim (b, x+ab)_1 \tag{1}$$

for all $a,b,x \in F$.

The previous description of B_q is also a definition of an incidence graph in the following way. The vertices with subscript 0 are the set P declared as points and the vertices with subscript 1 forms the set L declared as lines. The incidence structure (P, L) is called a biaffine plane [7, 13, 14]. We consider the points of an affine plane and all except one parallel class of its line. A point and a line are said to be incidence if they are adjacent. The incidence graph of a biaffine plane is clearly a bipartite graph with even girth at least 6. Alternatively, a biaffine plane can be described as what is left after removing from a projective plane all the lines through a given point and all points on one such line (the line at infinity).

In what follows we will consider the analytic description of B_q.

Our goal is to find a method how to extend B_q by just adding new edges within the set V_0 and within V_1 to obtain a vertex-transitive graph of diameter 2. For any $a, b \in F$ we let $M_0(a) = \{(a, x)_0;\ x \in F\}$ and $M_1(b) = \{(b, y)_1;\ y \in F\}$. We will add edges only within individual sets of the form $M_0(a)$ for $a \in F$, and within $M_1(b)$ for $b \in F$. We will say that a graph Γ is a *clustered extension* of B_q if Γ contains B_q as a spanning subgraph and the vertex set of each connected component of $\Gamma \backslash E(B_q)$ is a subset of $M_0(a)$ and $M_1(b)$ for $a, b \in F$. Our aim is the following:

Find all vertex-transitive clustered extensions Γ of B_q of diameter 2.

We show that the solution of the problem can be pinned down to finding certain very specific elements, subsets, and automorphisms of the field F.

Theorem 1 *Let $q = p^n$ be a prime power such that $q > 5$. The following condition $(*)$ is sufficient for the existence of a vertex-transitive clustered extensions Γ of B_q of such that Γ has diameter 2 and degree $q + \delta$ with $(q-1)/2 \le \delta \le q-1$:*

$(*)$ *There exists a non-zero element $t \in F$, a subset $C \subset F\backslash\{0\}$, and an automorphism σ of F, such that $|C| = \delta$, $C = -C$, $C \cup tC^{\sigma} = F\backslash\{0\}$, and $tt^{\sigma}C^{\sigma^2} = C$.*

Moreover, if $(n, p) = 1$, then the condition $(*)$ is also necessary for the existence of a vertex-transitive clustered extension with the above properties.

This result extends the results of [7] devoted to the McKay-Miller-Širáň graphs and generalizes some of the results of [12]. It allows for new interesting constructions, but also places severe restrictions on the ways clustered extensions can be constructed.

We also note, how Γ arises as a regular lift of a dipole. At the end we discuss applications for the constructions in [2], [7], some results of [12] and discuss existence of other generating sets satisfying the strong conditions of Theorem 1.

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Recent progress on Frobenius graphs and related topics

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Abstract

I will survey recent results on Frobenius graphs as models for interconnection networks.

Keywords: Cayley graph, Circulant graph, Frobenius graph, Gossiping, Minimum gossip time, Routing, Edge-forwarding index, Broadcasting time, Gaussian network, Eisenstein-Jacobi network.

Introduction

Searching for 'good' graphs to model interconnection networks is an ongoing endeavor in theoretical computer science and communication [4]. Roughly speaking, Cayley graphs are favored due to their strong fault-tolerance, high expansion in some cases, and vertex-symmetry that allows uniform routing at all vertices, among many other desirable properties. So far many classes of Cayley graphs (and circulant graphs in particular) have been proposed [4] by considering various invariants that measure performance of a network. In [9] it is proved that, as far as routing and gossiping are concerned, a large class of arc-transitive Cayley graphs, called the first kind Frobenius graphs [1, 9], are very attractive candidates in the following sense: any first kind Frobenius graph achieves minimum possible edge-forwarding index and gossiping time and possesses several other attractive routing and gossiping properties. In [2] it is proved further that the so-called second kind Frobenius graphs also have attractive routing and gossiping properties. I will talk about these results and two interesting families [6, 7, 8] of first kind Frobenius circulant graphs.

Searching for 'good' graphs is also motivated by the need of constructing perfect codes. In recent years, two families of graphs under the name of Gauss and Eisenstein-Jacobi are proposed [5] from a coding theoretic point of view. Gaussian networks are defined as certain Cayley graphs on the quotient rings of the ring of Gaussian integers, and Eisenstein-Jacobi networks are defined in terms of the quotient rings of the ring of Eisenstein-Jacobi integers. Among other things these two families of graphs have been found useful in constructing perfect codes [3, 5]. I will discuss recent results pertaining to these two families of graphs and connections between them and the above-mentioned Frobenius circulant graphs.

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