A definition of polyhedron

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This paper is a translation from a paper in french of 2002. The translation is due to Samuel Verbiese and Francis Buekenhout.

Introduction

The definitions given here are inspired by work of Jacques Tits between 1954 and 1962 devoted to incidence geometries, to buildings and to their relationships with groups, but these notions and relationships are not supposed to be known and are not necessary for the understanding of what follows.

The presentation seeks to remain as elementary as possible.

The motivations are to be found in various "polyhedral talks" performed by Edmond Dony and myself.

In my opinion, it is not presently possible to give a simpler definition of polygons and polyhedra without being exposed to fundamental errors.

It is obvious that this presentation is not adapted to the education of young pupils.

The historical evolution of definitions due to Euclid, Legendre and others is a captivating complement which I will treat in another text.

Definition of a polygon

This definition relies on three primitive (or primary) notions which are those of vertex, edge and incidence relation These notions are submitted to conditions or axioms.

(P) A polygon is constituted by the data of two non empty disjoint sets V and A and of a relationship I which is a subset of the product V x A.
The elements of V are called **vertices**.
The elements of A are calles **edges**.
I is said to be the **incidence relation** and its elements are called **chambers**.
If (s,a) is an element of I, one says that "s is incident to a", that "a is incident to s and that " s and a are incident".
These data are submitted to the following conditions :

(A1) Every vertex is incident to two edges
(A2) Every edge is incident to two vertices
(A3) Connectedness: for every vertex s and every edge a, there exists a finite sequence

$$s = x_1, x_2, ..., x_n = a$$

in which every pair of consecutive elements are incident.

Comments

1. The names "vertex" and "edge" are not so important. We need them in the expression of the above statements. We could replace them by "chair" and " table" without modifying the structure of a polygon. In other terms, we are not interested in the nature of vertices and edges.

2. If one weakens (A1) and (A2) in requiring that all vertices and all edges be incident to at most two elements, one opens the door to pre-polygons that include the polygons and the "polygonal paths" of elementary geometry.

3. Vertices and edges play the same role. There is a principle of duality. Every polygon (V, A, I) possesses a dual (A, V, I*) in which (a,s) is an element of I* if and only if (s, a) is an element of I.

Definition

Let (V, A, I) and (V', A', I') be two polygons. An **isomorphism** of the first on the second is a pair of bijections of V on V' and of A on A' which transforms every chamber of I into a chamber of I'. The converse property, i.e. that every chamber of I' is the image of a chamber of I, is automatic. As usual, an isomorphism of (V, A, I) on itself is called an **automorphism**.

Theorems

1. For every natural number $n \geq 2$ and for n countably infinite, there exists a polygon having n vertices.

2. Two polygons are isomorphic if and only if they have the same number of vertices.

3. If (V, A, I) is a polygon in which (s, a) and (s', a') are two chambers, there exists one and only one automorphism of (V, A, I) transforming (s, a) in (s', a'). Every polygon is **regular**.

4. The group of automorphisms of a polygon of n vertices is the dihedral group of order 2n.

Comments

1. The proofs are neither trivial, nor very difficult. We deal with them during the oral presentation.

2. We observe the appearance of two rather non-classical polygons. On the one side, the digon or polygon with two vertices, which we can however watch on every edge of every polyhedron. On the other side, the infinite polygon which is nothing else than the straight line on the integer numbers.

Embedded polygon

Let E be the (Euclidean) plane, the (Euclidean) space or any other space in which the following definition has a meaning. A polygon (V, A, I) is said to be **embedded** in E or it is a **polygon of E** if V is a set of points of E and if A is a set of closed segments of E such that every endpoint of an edge is a vertex. The polygon is said to be **regular** if each of its automorphisms extends to an automorphism of E.

Definition of a polyhedron

This definition relies on four primitive (or primary) notions which are those of vertex, edge, face and incidence relation. These notions are submitted to conditions or axioms.

(P) A polyhedron is constituted by the data of three non-empty disjoint sets V, A and F and of a relationship I which is a subset of the product V x A x F.
The elements of V are called **vertices**.
The elements of A are called **edges**.
The elements of F are called **faces**.
I is said to be the **incidence relation** and its elements are called **chambers**.
If x and y are two elements (vertices, edges or faces) of some chamber, x and y are said to be **incident**.
If (s,a) is an element of I, one says that "s is incident to a", that " a is incident to s" and that " s and a are incident".
A **wal**l is any pair of incident elements.
These data are submitted to the following conditions :

(A1) For every vertex s, the set of edges and faces incident to s provided with the incidence relation induced by I is a polygon.

(A2) For every face f, the set of edges and vertices incident to f provided with the incidence relation induced by I is a polygon.

(A3) For every edge a, the set of vertices and faces incident to a provided with the incidence relation induced by I is a polygon.

(A4) Connectedness: for every vertex s and every face f, there exists a finite sequence

$$s = x_1, x_2, ..., x_n = f$$

in which every pair of consecutive elements are incident.

Comments

1. The names "vertex", "edge" and "face" are not very important. We need them in the expression of the above statements. We could replace them by "chair", "table" and "mug" without affecting the structure of a polyhedron. In other terms, we are not interested in the nature of vertices, edges and faces.

2. If one loosens (A1) and (A2) in requiring that all vertices and all faces be incident to a pre-polygon and one proceeds similarly in (A3), one opens the door to pre-polyhedra that include the nets of polyhedra of elementary geometry.

3. Vertices and faces play the same role. There is a principle of duality. Every polyhedron (V, A, F, I) admits a dual (F, A, V, I).

4. It is interesting to modify (A3) and replace the word digon in it by the word polygon. One then obtains a triality principle. The structured sets defined in this way are called " thin geometries of rank three".

Definitions

Let (V, A, F, I) and (V', A', F', I') be two polyhedra. An **isomorphism** of the first onto the second is a trio of one to one mappings of V on V', of A to A' and of F to F' which map every chamber of I onto a chamber of I'. The converse property, i.e. that every chamber of I' is the image of a chamber of I, is automatic. As usual, an isomorphism of (V, A, F, I) onto itself is called an automorphism.

A polyhedron is said to be regular if for every chamber c and every chamber c' there exists an automorphism mapping c onto c'.

Theorems

1. Every wall is contained in two chambers.

2. If one decides that two chambers are adjacent if they have a common wall, the set of chambers is connected under the adjacency relation.

3. If c and c' are two chambers, there exists at most one automorphism mapping c onto c'.

4. In a regular polyhedron, every wall is invariant under one and only non-identity automorphism and the latter is of order two.

Comments

1. The proofs are neither trivial, nor very difficult. We keep them for the oral presentation.

2. One must not expect an exhaustive classification of the polyhedra as was the case for polygons. The world of polyhedra is wild. One must not even expect an exhaustive classification of the regular polyhedra : that world is wild too. It is possible to define covers of polyhedra and to classify the regular simply connected polyhedra.

3. One could express "convexity" using Euler’s formula as an axiom. In the usual practice, this is what is almost always done in an implicit way.

Embedded polyhedron

Let E be the (Euclidean) plane, the (Euclidean) space or any other space in which the following definition has a meaning. A polyhedron (V, A, F, I) is called **embedded** in E or it is a **polyhedron of E** if V is a set of points of E and if A is a set of closed segments of E such that every end of an edge is a vertex. An element of F is a polygon embedded in E all of whose vertices are in V and whose edges are in A. The polyhedron is said to be **regular in E** if it is regular and if each of its automorphisms extends to an automorphism of E.

Comments

1. It is possible to develop a theory of nets of polyhedra resulting in a beautiful general theorem.

2. It is possible to avoid the repetitions we have made separating the polygons and the polyhedra. There is a unified and generalized definition of the polytopes and of the thin geometries. As to Group Theory this matter is close to the Coxeter groups.

3. We just saw that a theory of polyhedra via incidence geometry and without topology is well developed. The present exception is the treatment of the generalized Euler formula which remains dependant on topology, perhaps temporarily.

4. It may happen that a polyhedron of E be regular as a

polyhedron but that it is not regular in E. Let's think of a rectangular parallelipiped. There are other more spectacular counter-examples.