The Tangramoid as a Mathematical Polyhedron

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Abstract

Tangramoids are families of 3D objects introduced by Samuel Verbiese [1]. One of these objects is called the tangramoid T. Our purpose is to provide an accurate mathematical description of it and to prove that T is a polyhedron in a sense due to Buekenhout.

1. Introduction

We start from views of Samuel Verbiese expressed in [1]. On the basis of the classical Tangram of seven pieces in its square shape, he saw objects in 3D leading to notions of Tangramoid. One of these has emerged due to peculiar features and we call it the tangramoid T. Here, we forget about its origin and describe it entirely, providing its mathematical existence as well. A second purpose deals with the structure of the tangramoid. We prove that T is a polyhedron in the sense of Buekenhout. This structure is redefined in Section 2 in order to make the needed mathematics self-contained. In our description and study of T, a crucial role is played by its symmetry group, a rotation group of order 4 in 3D.

2. Polyhedron in the sense of Buekenhout

We recall definitions from Buekenhout [3]. Most ideas derive from the work of Jacques Tits during the period 1954-1962 when he got his deep theory of buildings. A historical account is provided in Buekenhout [4] and a geometrical context can be found in [5].

<u>Definition of polygon</u>
This definition relies on three primitive (or primary) notions which are those of vertex, edge and incidence relation. These notions are submitted to conditions or axioms.

(P) A polygon is constituted by the data of two non-empty disjoint sets V and E and of a relationship I which is a subset of the cartesian product V x E.
Here the elements of V are called **vertices**. Any other name may be used.
Here the elements of E are called **edges**. Any other name may be used.
I is said to be the **incidence relation** and its elements are called **chambers**.
If (v,e) is an element of I, it is said that "v is incident to e", that "e is incident to v" and that "v and e are incident".
These data are submitted to the following conditions :

(A1) Every vertex is incident to two edges
(A2) Every edge is incident to two vertices
(A3) **Connexity** : for every vertex v and every edge e, there exists a finite sequence

$$v = x_1, x_2, ..., x_n = e$$

in which every pair of consecutive elements are incident.

<u>Embedded polygon</u>
Let S be the (Euclidean) plane, the (Euclidean) space or any other space in which the following definition is meaningful. A polygon (V, E, I) is said to be **embedded** in S or it is a **polygon of S** if V is a set of points of S and if E is a set of closed segments of S such that every endpoint of an edge is a vertex incident to it.

<u>Definition of polyhedron</u>
This definition relies on four primitive (or primary) notions which are those of vertex, edge, face and incidence relation. These notions are submitted to conditions or axioms.

(P) A polyhedron is constituted by the data of three non-empty disjoint sets V, E and F and of a relationship I which is a subset of the product V x E x F.
Here the elements of V are called **vertices**.
Here the elements of E are called **edges**.
Here the elements of F are called **faces**.
I is said to be the **incidence relation** and its elements are called **chambers**.
If x and y are two elements (vertices, edges or faces) of some chamber, x and y are said to be **incident**.
If (v,e) is an element of I, it is said that "v is incident to e", that " e is incident to v" and that " v and e are incident".
These data are submitted to the following conditions :

(A1) For every vertex v, the set of edges and faces incident to v having the incidence relation induced by I is a polygon. This polygon is called the **Residue of v**. It is the classical " Vertex figure" due to Schläfli.
(A2) For every face f, the set of edges and vertices incident to f having the incidence relation induced by I is a polygon. This polygon is called the **Residue of f**.
(A3) For every edge e, every vertex incident to e and every face incident to e are incident. In other words, the residue of an edge is a **digon**.
(A4) **Connexity** : for every vertex v and every face f, there exists a finite sequence

$$v = x_1, x_2, ..., x_n = f$$

in which every pair of consecutive elements are incident.

<u>Embedded polyhedron</u>
Let S be the (Euclidean) plane, the (Euclidean) space or any other space in which the following definition is meaningful. A polyhedron (V, E, F, I) is said to be **embedded** in S or it is a **polyhedron of S** if V is a

set of points of S, E is a set of closed segments of S such that every endpoint of an edge is a vertex incident to it and F is a set of polygons embedded in S whose vertices and edges are members of V and E.

3. Construction of the tangramoid T

We construct the tangramoid T from the regular octahedron Oc. First, we deal with the vertices of T, then with the edges, the faces and eventually with the incidence relation. The construction may be followed on Figure 1 where a neat physical and necessary picture is provided by the father of T.

3.1 Vertices

The tangramoid T is related to the regular octahedron Oc, and we start the construction of T from Oc. Let 1, $\underline{1}$, 2, $\underline{2}$, 3 and $\underline{3}$ be Oc's six vertices, where 1, $\underline{1}$ are opposite as well as 2, $\underline{2}$ and 3, $\underline{3}$. The vertices 1, $\underline{1}$ subsequently play a particular role and we imagine them on a "vertical line", 1 on top and $\underline{1}$ down. The straight line 1$\underline{1}$ is the axis of a rotation group Z_4 which is the symmetry group of the construction that will be performed and the symmetry group of T.

A generating rotation of this group can be seen as r = (1)($\underline{1}$)(2,3,$\underline{2}$,$\underline{3}$). This notation expresses the fact that 1 and $\underline{1}$ are fixed points under r, that r(2)=3, r(3)=$\underline{2}$, r($\underline{2}$)=$\underline{3}$ and r($\underline{3}$)=2. The group Z_4 consists of r, r^2, r^3 and r^4=I. It helps to visualize 2,3,$\underline{2}$,$\underline{3}$ as consecutive points of a square on the "equator".

Next we need to select some middle points of edges from Oc : so let us first select the middle point m(23) on the "equator" of Oc and the middle point m(12) on the upper edge 12 of Oc. Observe that in our notation m(23) = m(32). Also, a transform or image under Z_4 of a middle point is a middle point as well, and so r(m(23)) = m(3$\underline{2}$) is a middle point we need. Together with the images of these two middle points under Z_4, we thus add eight more vertices to those of Oc.

Next, let us consider the "left" triangle 2m(23)m(12). We introduce a corresponding vertex 2' which we see as "interior" to Oc such that the three lines 2'2, 2'm(23) and 2'm(12) be orthogonal. Physically we see 2' at a distance one fourth on the segment [2$\underline{2}$], from 2 on. We observe that the vertices m(23), 2' and m(2$\underline{3}$) are collinear and that the line m(12)2' is "vertical". With vertex 2' and its images we thus add four vertices. A visual observation is that T becomes "non convex" and even "chiral".

Finally, let us consider the triangle 1m(12)m(13). We introduce a corresponding vertex called m(23)' that we see "exterior" to Oc such that the lines m(23)m(23)', m(12)m(23)' and m(13)m(23)' be orthogonal. In particular m(23)' is on the "vertical" of m(23). With the vertex m(23)' and its Z_4-images we thus add four last vertices, which, we observe, form a square for which m(12) and its Z_4-images are the middle points of the edges. Thus we dispose of a set V of 22 vertices which are listed now from top to bottom for later use:

1
m(12) m(13) m(1$\underline{2}$) m(1$\underline{3}$)
m(23)' m(3$\underline{2}$)' m($\underline{2}\underline{3}$)' m($\underline{3}$2)'
2, 3, $\underline{2}$, $\underline{3}$
m(23) m(3$\underline{2}$) m($\underline{2}\underline{3}$) m($\underline{3}$2)
2' 3' $\underline{2}$' $\underline{3}$'
$\underline{1}$

3.2 Edges

Let us proceed bottom up this time, starting with Oc and the vertices we just described.
First,
$\underline{1}$2 $\underline{1}$3 $\underline{1}\underline{2}$ $\underline{1}\underline{3}$ are the 4 lower edges of T, all issued from $\underline{1}$.
Next,
2m(23) 3m(3$\underline{2}$) $\underline{2}$m($\underline{2}\underline{3}$) $\underline{3}$m($\underline{3}$2) and
m(23)3 m(3$\underline{2}$)$\underline{2}$ m($\underline{2}\underline{3}$)$\underline{3}$ m($\underline{3}$2)2 are 8 edges of T on the Oc-"equator".

Then
22’ 33’ $\underline{22}$’ $\underline{33}$’ and
2’m(23) 3’m(3$\underline{2}$) $\underline{2}$’m($\underline{23}$) $\underline{3}$’m($\underline{3}$2) are 8 edges of T in the “equatorial “ plane of Oc.
Now,
2m(12) 3m(13) $\underline{2}$m(1$\underline{2}$) $\underline{3}$m(1$\underline{3}$) and
2’m(12) 3’m(13) $\underline{2}$’m(1$\underline{2}$) $\underline{3}$’m(1$\underline{3}$) and
m(23)m(12) m(3$\underline{2}$)m(13) m($\underline{23}$)m(1$\underline{2}$) m($\underline{3}$2)m(1$\underline{3}$) are 12 edges up from the “equatorial plane”.
Also,
m(12)m(13) m(13)m(1$\underline{2}$) m(1$\underline{2}$)m(1$\underline{3}$) m(1$\underline{3}$)m(12) and
m(12)m(23)’ m(13)m(3$\underline{2}$)’ m(1$\underline{2}$)m($\underline{23}$)’ m(1$\underline{3}$)m($\underline{3}$2)’ and
m(23)’m(13) m(3$\underline{2}$)’m(1$\underline{2}$) m($\underline{23}$)’m(1$\underline{3}$) m($\underline{3}$2)’m(12) are 12 edges of T in a “horizontal plane”.
Finally,
m(23)’1 m(3$\underline{2}$)’1 m($\underline{23}$)’1 m($\underline{3}$2)’1 are the 4 top edges, all on 1.
Thus we get the 48 edges of T.

3.3 Faces
Again, we proceed bottom up.
First we select
$\underline{1}$2m(23)3 $\underline{1}$3m(3$\underline{2}$)$\underline{2}$ $\underline{12}$m($\underline{23}$)$\underline{3}$ $\underline{13}$m($\underline{3}$2)2 which are the 4 lower faces. They are quadrangles degenerated in triangles.
Next,
22’m(23) 33’m(3$\underline{2}$) $\underline{22’}$m($\underline{23}$) $\underline{33’}$m($\underline{3}$2) are 4 triangular faces in the “equatorial plane” of Oc.
Then,
22’m(12) 33’m(13) $\underline{22}$’m(1$\underline{2}$) $\underline{33}$’m(1$\underline{3}$) and
m(23)2’m(12) m(3$\underline{2}$)3’m(13) m($\underline{23}$)$\underline{2}$’m(1$\underline{2}$) m($\underline{32}$)$\underline{3}$’m(1$\underline{3}$) are 8 “vertical” faces.
Next,
m(23)m(12)m(13)3 m(3$\underline{2}$)m(13)m(1$\underline{2}$)$\underline{2}$ m($\underline{23}$)m(1$\underline{2}$)m(1$\underline{3}$)$\underline{3}$ m($\underline{3}$2)m(1$\underline{3}$)m(12)2 are 4 quadrangles, actually rhombs because each one is the union of two equilateral triangles with a common edge in a face of Oc.
Also,
m(12)m(23)’m(13) m(13)m(3$\underline{2}$)’m(1$\underline{2}$) m(1 $\underline{2}$)m($\underline{23}$)’m(1$\underline{3}$) m(1 $\underline{3}$)m($\underline{3}$2)’m(12) are 4 triangular faces of T. They are “horizontal”.
Finally,
m(23)’m(13)m($\underline{3}$2)'1 m(3$\underline{2}$)’m(1$\underline{2}$)m($\underline{23}$)'1 m($\underline{23}$)’m(1$\underline{3}$)m($\underline{3}$2)'1 m($\underline{3}$2)’m(12)m(23)'1 are 4 quadrangular faces degenerated in triangles for the “roof” of T.
We are getting 28 faces.
Thus T satisfies Euler’s formula but we still need to check that it is indeed a polyhedron. Courage is provided by the idea that you may blow on T from the “inside” and see it on a sphere.

3.4 Incidence relation
The set of 22 vertices, 48 edges and 28 faces requires an incidence relation which is very simply defined by inclusion. A vertex v and an edge e are incident if v is an end of e. For instance 1 is incident to 1m(23)'. An edge e is incident to a face f if e is actually an edge of f. For instance m(12)2' is incident to m(23)m(12)2'. A vertex v is likewise incident to a face f if v is incident to f. This is ending the construction of T in terms of vertices, edges, faces and incidence.

4. Main result

We want to prove the

Theorem
The tangramoid T is a polyhedron in the sense of Section 2.

Proof
This needs to be followed on our symbolic model, not on Figure 1, in order to ensure that everything fits.

Step 1: every edge is incident with two vertices. This is obvious from the construction.
Step 2: every edge is incident with two faces. This requires a series of checks using the list of edges from 3.2.
2.1 The edge 21 is incident to the quadrangles 213m(23) and 213m(32).
2.2 The edge m(23)3 is incident to the quadrangle m(23)312 and to the rhomb m(23)3m(13)m(12).
2.3 The edge m(23)m(12) is incident to the rhomb m(23)3m(13)m(12) and to the triangle m(23)m(12)2'.
2.4 The edge 3m(13) is incident to the rhomb m(23)3m(13)m(12) and to the triangle 3m(13)3'.
2.5 The edge m(12)m(13) is incident to the rhomb m(23)3m(13)m(12) and to the triangle m(12)m(23)'m(13).
2.6 The edge m(23)'1 is incident to the quadrangles m(23)'1m(21)2 and m(23)'1m(31)3.
Step 3: The residue of every face is connected. This is obvious since those residues are either triangles or quadrangles.
Step 4: The residue of every vertex is connected and so it is a polygon as required in view of Step 2. Indeed the residues of vertices 1, m(12), m(23)’, 2, m(23), 2’, 1 are respectively polygons of 4, 7, 3, 5, 4, 3, 4 vertices. This statement requires straightforward work whose details are not written down. That statement is a major part of the structure of T.
Step 5: The full set of vertices, edges and faces is connected, namely, property (A4) of Section 2 holds. We can check that vertex 1 is connected to all other vertices and so to all edges and faces in view of their descriptions thanks to vertices.
QED

It clearly appears also that T is an Embedded Polyhedron in the Euclidean space of dimension 3.

5. To the Tangram

The simple structure for T chosen so far does not entirely satisfy the original view of Verbiese on the tangramoid, especially the relation to the classical Tangram puzzle of seven pieces [2] in its square shape : the four rhombic faces must be opaque because the projection allows for only one square and one parallelogram.

Let us observe also that T is a chiral figure in 3D space.

Conclusion

This work meets, for the case of the closed tangramoid, the hope expressed in [1] that thorough mathematical developments could be made to establish that tangramoids created in the 'sphere of art' also truly belong to the 'sphere of mathematics'. The second author feels sincerely indebted to the first for having spearheaded this important contribution.

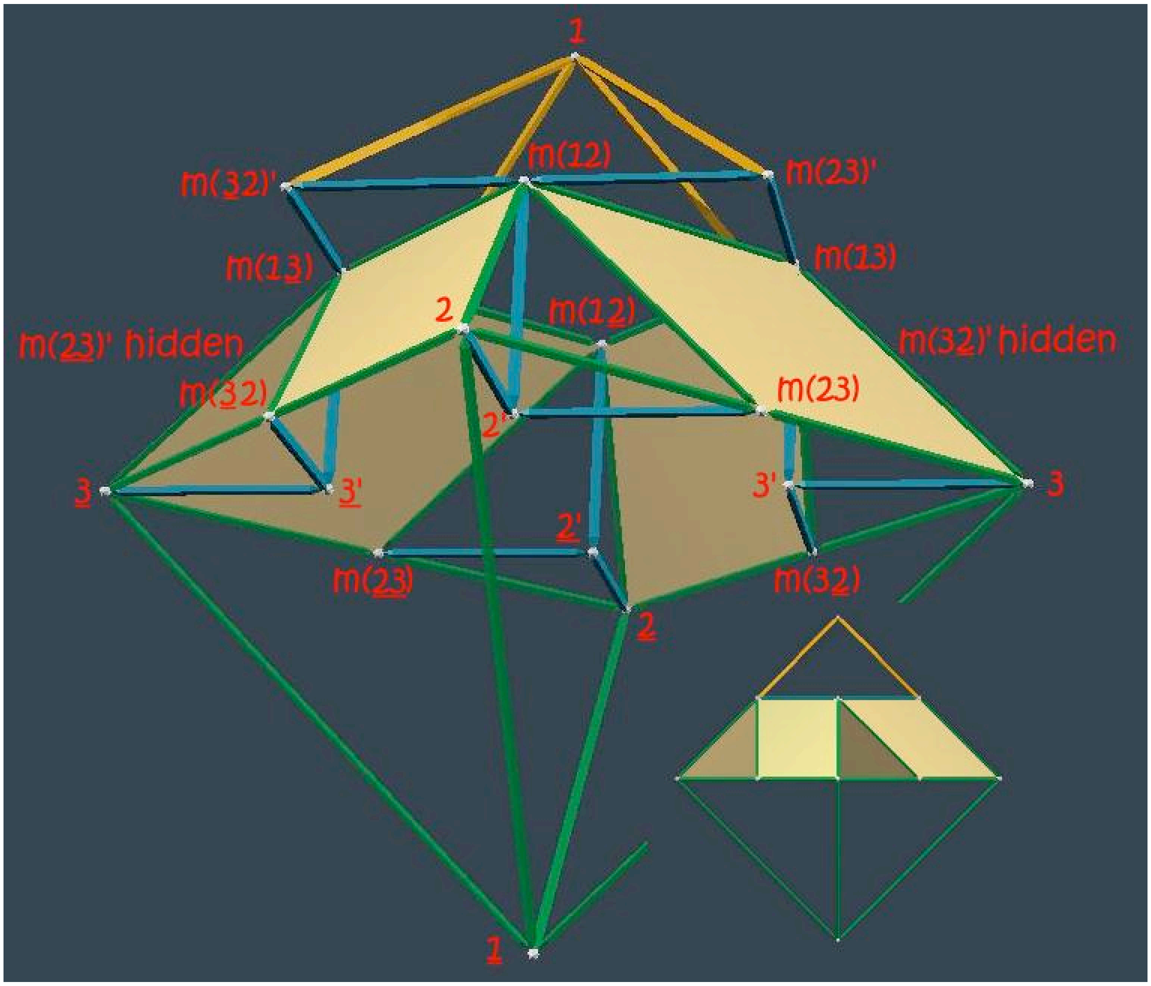

Figure 1: *The vertices, edges and opaque rhombs of the tangramoid T generated from the regular octahedron Oc, rendered after Zometool parts [6]in Scott Vorthmann's vZome modeling software [7] with tiny connectors and annotated with the herein used notations,*
and, in insert, one of the four "Tangram projections" of the polyhedron T with opaque rhombs

References

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