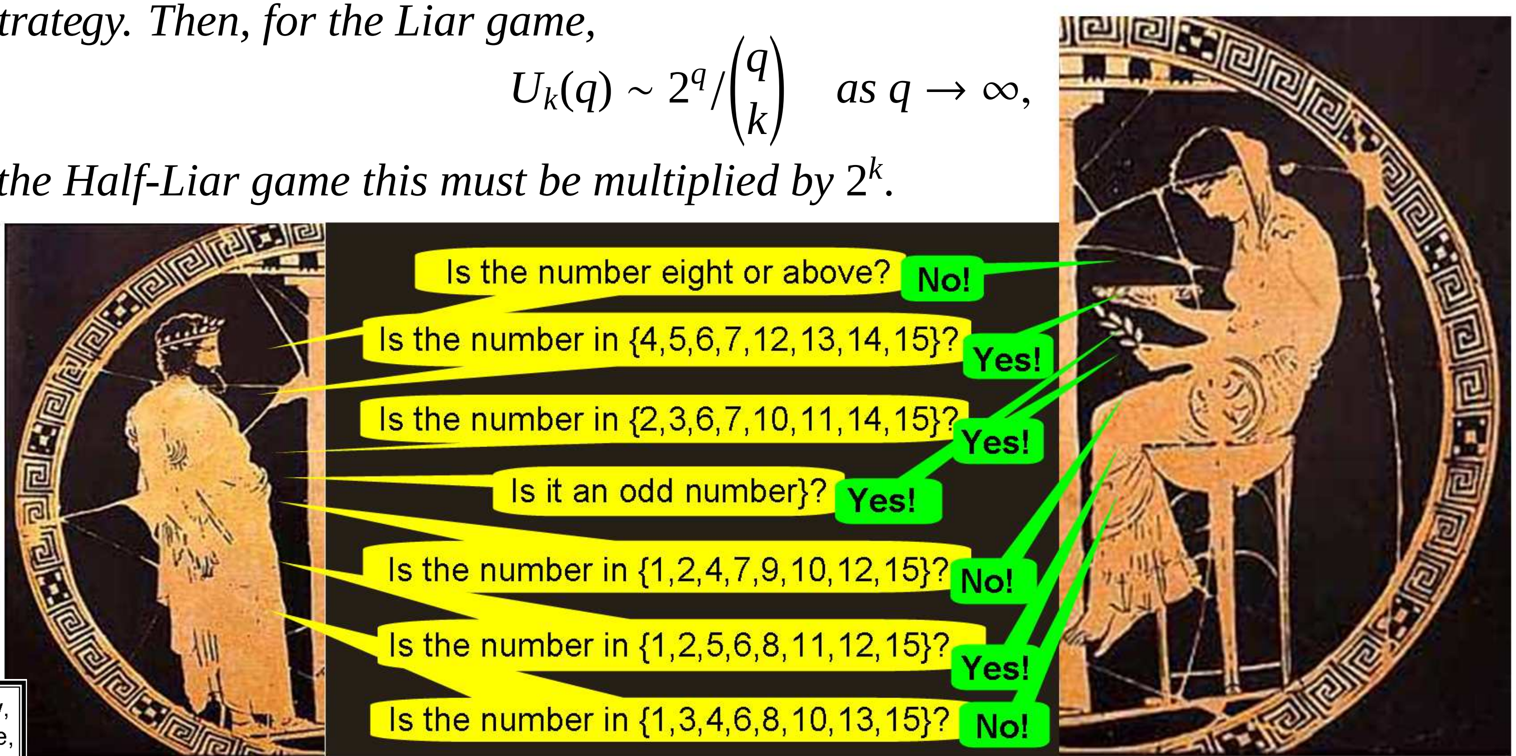
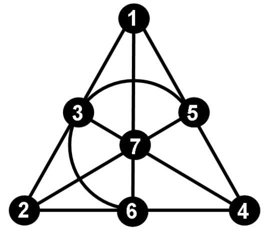
The Asymptotic (Half) Liar Formula *Carole and Paul play the following 'Liar' game: Carole picks a value $x \in \{0, \ldots, n\}$; Paul wins if he identifies x by asking no more than q Yes-No questions of the form "Is x in subset X?"; Carole may lie at most k times, in trying to prevent this. In the 'Half-Liar' game, her lies may only be Yes's. Denote by $U_k(q)$ the maximum value of n for which Paul has a guaranteed winning strategy. Then, for the Liar game,*

$$U_k(q) \sim 2^q / \binom{q}{k} \quad \text{as } q \to \infty,$$

while for the Half-Liar game this must be multiplied by 2^k.

If Paul can ask the $q = 7$ Y/N questions shown on the right, and if Carole is allowed to lie at most $k = 1$ times, then Paul can always identify her x value for $n = 15$ (so that $U_1(7) \geq 15$).





Lines:

1 2 3   1 6 7   1 4 5
2 4 6   2 5 7   3 4 7
3 5 6

> Half the world's folly,
> And sorrow and woe,
> Comes from a Yes
> That should be a No!
>
> Traditional

A linear error-correcting code based on the Fano plane above will identify any lie and the correct value of x. Firstly, how to detect a lie: (1) note which questions get a Y answer. If there are three or fewer Ys proceed as follows; otherwise interchange the roles of Y and N in what follows. (2) If there are zero Ys, or three Ys whose positions form a line in the Fano plane, there are no lies; one Y must always be a lie; for two Y positions there will be a unique third position forming a Fano plane line and this is a lie; finally, for three Ys *not* on a line, use the four N positions: three will form a line and the fourth is the lie position. Once Carole's lie, if she made one, is corrected, we may write the correct sequence of Ys and Ns as a binary string of length 7 (Y=1, N=0). Take the first four positions: they encode a binary number between 0 (i.e., 0000) and 15 (i.e., 1111). This is x.

In 1992, Joel Spencer gave the complete asymptotic solution to the Liar Game, popularised by Stanisław Ulam in the 70s. The beautifully simple extension to the Half Liar Game he found in collaboration with Ioana Dumitriu 10 years later.

Web link: www.math.tamu.edu/~catherine.yan/Files/Halflie.pdf. The perfect strategy for $n = 15$, $q = 7$ and $k = 1$, as illustrated here, is from www.maths.qmul.ac.uk/~pjc/slides/lmspop.pdf. The Delphic Oracle image is from en.wikipedia.org/wiki/Delphi.

Further reading: *Enumerative Combinatorics, Vol. 2* by R.P. Stanley, Cambridge University Press, 2001.